GRAPHS, GROUPS AND SELF-SIMILARITY

Ruy Exel and Enrique Pardo

We study a family of C*-algebras generalizing both Katsura algebras and certain algebras introduced by Nekrashevych in terms of self-similar groups.

1. Introduction.

The purpose of this paper is to give a unified treatment to two classes of C*-algebras which have been studied in the past few years from rather different points of view, namely Katsura's algebras [**7**], and certain algebras constructed by Nekrashevych [**10**], [**11**] from self-similar groups.

The realization that these classes are indeed closely related, as well as the fact that they could be given a unified treatment, came to our mind as a result of our earlier attempt [**3**] to understand Katsura's algebras $\mathcal{O}_{A,B}$ from the point of view of inverse semigroups. The fact, proven by Katsura in [**7**], that all Kirchberg algebras may be described in terms of his $\mathcal{O}_{A,B}$ was, in turn, a strong motivation for that endeavor.

While studying $\mathcal{O}_{A,B}$, it slowly became clear to us that the two matricial parameters A and B play very different roles. The reader acquainted with Katsura's work will easily recognize that the matrix A is destined to be viewed as the edge matrix of a graph, but it took us much longer to realize that B should be thought of as providing parameters for an action of the group $\mathbb{Z}$ on the graph given by A. In trying to understand these different roles, some interesting arithmetic popped up sparking a connection with the work done by Nekrashevych [**11**] on the C*-algebra $\mathcal{O}_{(G,X)}$ associated to a self-similar group (G, X).

While Nekrashevych's algebras contain a Cuntz algebra, Katsura's algebras contain a graph C*-algebra. This fact alone ought to be considered as a hint that self-similar groups lie in a much bigger class, where the group action takes place on the path space of a graph, rather than on a rooted tree (which, incidentally, is the path space of a bouquet of circles).

One of the first important applications of the idea of self-similarity in group theory is in constructing groups with exotic properties [**5**], [**6**]. Many of these are defined as subgroups of the group of all automorphisms of a tree. Having been born from automorphisms, it is natural that the theory of self-similar groups generally assumes that the group acts *faithfully* on its tree (see, e.g. [**11**: Definition 2.1]).

However, based on the example provided by Katsura's algebras, we decided that perhaps it is best to view the group on its own, the action being an extra ingredient.

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The main idea behind self-similar groups, namely the equation

$$g(xw) = yh(w) \tag{1.1}$$

appearing in [**11**: Definition 2.1], and the subsequent notion of *restriction*, namely

$$g|_x := h,$$

depend on faithfulness, since otherwise the group element h appearing in (1.1) would not be unique and therefore will not be well defined as a function of g and x. Working with non-faithful group actions we were forced to postulate a functional dependence

$$h = \varphi(g, x),$$

and we were surprised to find that the natural properties expected of φ are that of a group cocycle.

To be precise, the ingredients needed in our generalization of self-similar groups are: a countable discrete group G, an action

$$G \times E \to E$$

of G on a finite graph $E = (E^0, E^1, r, d)$, and a one-cocycle

$$\varphi : G \times E^1 \to G$$

for the action of G on the edges of E.

Starting with this data (satisfying a few other natural axioms) we construct an action of G on the space of finite paths E^* which satisfy the "self-similarity" equation

$$g(\alpha\beta) = (g\alpha)\big(\varphi(g,\alpha)\beta\big), \quad \forall\, g \in E, \quad \forall\, \alpha, \beta \in E^*.$$

Adopting a philosophy similar to that embraced by Katsura and Nekrashevych, we define a C*-algebra, denoted

$$\mathcal{O}_{G,E},$$

in terms of generators and relations inspired by the above group action. The study of $\mathcal{O}_{G,E}$ is, thus, the purpose of this paper.

Given a self-similar group (G, X), if we consider X as the set of edges of a graph with a single vertex, and if we define $\varphi(g, x) = g|_x$, then our $\mathcal{O}_{G,E}$ coincides with Nekrashevych's $\mathcal{O}_{(G,X)}$.

On the other hand, if we are given two integer $N \times N$ matrices A and B, we may form a graph E with vertex set $E^0 = \{1, 2, \ldots, N\}$ and with $A_{i,j}$ edges from vertex i to vertex j. We may then use B to define an action of $\mathbb{Z}$ on E, by fixing all vertexes and acting on the set of edges as follows: denote the edges in E from i to j by $e_{i,j,n}$, where $0 \leq n < A_{i,j}$. Given $m \in \mathbb{Z}$, we perform the Euclidean division of $mB_{i,j} + n$ by $A_{i,j}$, say

$$mB_{i,j} + n = \hat{k}A_{i,j} + \hat{n}$$

with $0 \leq \hat{n} < A_{i,j}$, and put

$$\sigma_m(e_{i,j,n}) = e_{i,j,\hat{n}},$$

so that the group element m permutes the $A_{i,j}$ edges from i to j in the same way that addition by $mB_{i,j}$, modulo $A_{i,j}$, permutes the integers $\{0, 1, \ldots, A_{i,j} - 1\}$.

The quotient $\hat{k}$ on the above Euclidean division also plays an important role, being used in the definition of the cocycle

$$\varphi(m, e_{i,j,n}) = \hat{k}.$$

In possession of the graph, the action of $\mathbb{Z}$, and the cocycle φ, we apply our construction and find that $\mathcal{O}_{G,E}$ is isomorphic to Katsura's $\mathcal{O}_{A,B}$.

So, both Nekrashevych's and Katsura's algebras become special cases of our construction. We therefore believe that the project of studying such group actions on path spaces as well as the corresponding algebras is of great importance.

Taking the first few steps we have been able to describe $\mathcal{O}_{G,E}$ as the C*-algebra of an étale groupoid $\mathcal{G}_{G,E}$, whose construction is remarkably similar to the groupoid associated to the relation of "tail equivalence with lag" on the path space, as described by Kumjian, Pask, Raeburn and Renault in [**8**].

The first similarity is that our groupoid $\mathcal{G}_{G,E}$ has the exact same unit space as the corresponding graph groupoid, namely the infinite path space. The second, and most surprising similarity is that $\mathcal{G}_{G,E}$ is also described by a *lag* function, except that the values of the lag are not integer numbers, as in [**8**], but lie in a slightly more complicated group, the semi-direct product of the corona group of G by the right shift automorphism (see below for precise definitions).

The techniques we use to give $\mathcal{O}_{G,E}$ a groupoid model bear heavily on the theory of tight representations of inverse semigroups developed by the first named author in [**2**]. In particular, from our initial data we construct an abstract inverse semigroup $\mathcal{S}_{G,E}$ and show that $\mathcal{O}_{G,E}$ is the universal C*-algebra for tight representations of $\mathcal{S}_{G,E}$.

As a second step we again take inspiration from Nekrashevych [**10**] and give a description of $\mathcal{O}_{G,E}$ as a Cuntz Pimnsner algebra for a very natural correspondence M over the algebra

$$C(E^0) \rtimes G.$$

As a result we are able to prove that $\mathcal{O}_{G,E}$ is nuclear when the G is amenable.

We would like to stress that, like Nekrashevych [**11**: Theorem 5.1], our groupoid $\mathcal{G}_{G,E}$ is constructed as a groupoid of germs. However, departing from Nekrashevych's techniques, we use Patterson's [**12**] notion of "germs", rather than the one employed in [**11**: Section 5]. While agreeing in many cases, the former has a much better chance of producing Hausdorff groupoids and, in our case, we may give a precise characterization of Hausdorffness in terms of a property we call residual freeness (see below for the precise definition).

Part of this work was done during a visit of the second named author to the Departamento de Matemática da Universidade Federal de Santa Catarina (Florianópolis, Brasil) and he would like to express his thanks to the host center for its warm hospitality.

2. Groups acting on graphs.

Let $E = (E^0, E^1, r, d)$ be a directed graph, where E^0 denotes the set of *vertexes*, E^1 is the set of *edges*, r is the *range* map, and d is the *source*, or *domain* map.

By definition, a *source* in E is a vertex $x \in E^0$, for which $r^{-1}(x) = \varnothing$. Thus, when we say that a graph has *no sources*, we mean that $r^{-1}(x) \neq \varnothing$, for all $x \in E^0$.

By an *automorphism* of E we shall mean a bijective map

$$\sigma : E^0 \dot\cup E^1 \to E^0 \dot\cup E^1$$

such that $\sigma(E^i) \subseteq E^i$, for $i = 0, 1$, and moreover such that $r \circ \sigma = \sigma \circ r$, and $d \circ \sigma = \sigma \circ d$, on E^1. It is evident that the collection of all automorphisms of E forms a group under composition.

By an action of a group G on a graph E we shall mean a group homomorphism from G to the group of all automorphisms of E.

If X is any set, and if σ is an action of a group G on X, we shall say that a map

$$\varphi : G \times X \to G$$

is a *one-cocycle* for σ, when

$$\varphi(gh, x) = \varphi\big(g, \sigma_h(x)\big)\varphi(h, x), \tag{2.1}$$

for all $g, h \in G$, and all $x \in X$. Plugging $g = h = 1$ above we see that

$$\varphi(1, x) = 1, \tag{2.2}$$

for every x.

2.3. Standing Hypothesis. Throughout this work we shall let G be a countable discrete group, E be a finite graph with no sources, σ be an action of G on E, and

$$\varphi : G \times E^1 \to G$$

be a one-cocycle for the restriction of σ to E^1, which moreover satisfies

$$\sigma_{\varphi(g,e)}(x) = \sigma_g(x), \quad \forall\, g \in G, \quad \forall\, e \in E^1, \quad \forall\, x \in E^0. \tag{2.3.1}$$

The assumptions that E is finite and has no sources will in fact only be used in the next section and it could probably be removed by using well known graph C*-algebra techniques.

By a *path* in E of *length* $n \geq 1$ we shall mean any finite sequence of the form

$$\alpha = \alpha_1\alpha_2 \ldots \alpha_n,$$

where $\alpha_i \in E^1$, and $d(\alpha_i) = r(\alpha_{i+1})$, for all i (this is the usual convention when treating graphs from a categorical point of view, in which functions compose from right to left). The *range* of α is defined by

$$r(\alpha) = r(\alpha_1),$$

while the *source* of α is defined by

$$d(\alpha) = d(\alpha_n).$$

A vertex $x \in E^0$ is considered to be a path of length zero, in which case we set $r(x) = d(x) = x$.

For every integer $n \geq 0$ we denote by E^n the set of all paths in E of length n (this being consistent with the already introduced notations for E^0 and E^1). Finally, we denote by E^* the sets of all finite paths, and by $E^{\leq n}$ the set of all paths of length at most n, namely

$$E^* = \bigcup_{k \geq 0} E^k, \quad \text{and} \qquad E^{\leq n} = \bigcup_{k=0}^{n} E^k.$$

We will often employ the operation of *concatenation* of paths. That is, if (and only if) α and β are paths such that $d(\alpha) = r(\beta)$, we will denote by $\alpha\beta$ the path obtained by juxtaposing α and β.

In the special case in which α is a path of length zero, the concatenation $\alpha\beta$ is allowed if and only if $\alpha = r(\beta)$, in which case we set $\alpha\beta = \beta$. Similarly, when $|\beta| = 0$, then $\alpha\beta$ is defined iff $d(\alpha) = \beta$, and then $\alpha\beta = \alpha$.

We would now like to describe a certain extension of σ and φ to finite paths.

2.4. Proposition. *Under the assumptions of (2.3) there exists a unique pair (σ^*, φ^*), formed by an action σ^* of G on E^* (viewed simply as a set), and a one-cocycle φ^* for σ^*, such that, for every $n \geq 0$, every $g \in G$, and every $x \in E^0$, one has that:*

(i) $\sigma^*_g = \sigma_g$, *on* $E^{\leq 1}$,
(ii) $\varphi^*(g, x) = g$,
(iii) $\varphi^* = \varphi$, *on* $G \times E^1$,
(iv) $\sigma^*_g(E^n) \subseteq E^n$,
(v) $r \circ \sigma^*_g = \sigma_g \circ r$, *on* E^n,
(vi) $d \circ \sigma^*_g = \sigma_g \circ d$, *on* E^n,
(vii) $\sigma_{\varphi^*(g,\alpha)}(x) = \sigma_g(x)$, *for all* $\alpha \in E^n$,
(viii) σ^*_1 *is the identity*[1] *on* E^n,
(ix) $\sigma^*_g(\alpha\beta) = \sigma^*_g(\alpha)\ \sigma^*_{\varphi^*(g,\alpha)}(\beta)$, *provided* α *and* β *are finite paths with* $\alpha\beta \in E^n$,
(x) $\varphi^*(g, \alpha\beta) = \varphi^*\big(\varphi^*(g, \alpha), \beta\big)$, *provided* α *and* β *are finite paths with* $\alpha\beta \in E^n$.

Proof. Initially notice that, once (v), (vi) and (vii) are proved, the concatenation of the paths "$\sigma^*_g(\alpha)$" and "$\sigma^*_{\varphi^*(g,\alpha)}(\beta)$", appearing in (ix), is permitted because

$$r\big(\sigma^*_{\varphi^*(g,\alpha)}(\beta)\big) \overset{\text{(v)}}{=} \sigma_{\varphi^*(g,\alpha)}(r(\beta)) \overset{\text{(vii)}}{=} \sigma_g(r(\beta)) = \sigma_g\big(d(\alpha)\big) \overset{\text{(vi)}}{=} d\big(\sigma^*_g(\alpha)\big).$$

[1] This is evidently already included in the statement that σ^* is an action, but we repeat it here to aid our proof by induction.

For every g in G, define σ^*_g on $E^{\leq 1}$ to coincide with σ_g. Also, define φ^* on $G \times E^{\leq 1}$ by (ii) and (iii). It is then clear that (i–iii) hold and it is easy to see that the remaining properties (iv–x) hold for all $n \leq 1$.

We shall complete the definitions of σ^* and φ^* by induction, so we assume that $m \geq 1$, that

$$\sigma^*_g : E^{\leq m} \to E^{\leq m}$$

is defined for all g in G, that

$$\varphi^* : G \times E^{\leq m} \to G,$$

is defined, and that (i–x) hold for all $n \leq m$. We then define

$$\sigma^*_g : E^{m+1} \to E^{m+1}$$

for all g in G, and

$$\varphi^* : G \times E^{m+1} \to G,$$

by induction as follows. Given $\alpha \in E^{m+1}$, write $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $\alpha'' \in E^m$, and put

$$\sigma^*_g(\alpha) = \sigma_g(\alpha')\sigma^*_{\varphi(g,\alpha')}(\alpha''), \quad \text{and} \quad \varphi^*(g,\alpha) = \varphi^*\big(\varphi(g,\alpha'),\alpha''\big). \tag{2.4.1}$$

A quick analysis, as done in the first paragraph of this proof, shows that the concatenation of "$\sigma_g(\alpha')$" and "$\sigma^*_{\varphi(g,\alpha')}(\alpha'')$", appearing above, is permitted. We next verify (iv–x), substituting $m+1$ for n.

We have that the length of $\sigma^*_g(\alpha)$, as defined above, is clearly $1+m$, thus proving (iv). With respect to (v) we have that

$$r\big(\sigma^*_g(\alpha)\big) = r\big(\sigma_g(\alpha')\big) = \sigma_g\big(r(\alpha')\big) = \sigma_g\big(r(\alpha)\big).$$

As for (vi), notice that

$$d\big(\sigma^*_g(\alpha)\big) = d\big(\sigma^*_{\varphi(g,\alpha')}(\alpha'')\big) = \sigma_{\varphi(g,\alpha')}\big(d(\alpha'')\big) = \sigma_g\big(d(\alpha'')\big) = \sigma_g\big(d(\alpha)\big).$$

Given $x \in E^0$, we have that

$$\sigma_{\varphi^*(g,\alpha)}(x) = \sigma_{\varphi^*(\varphi(g,\alpha'),\alpha'')}(x) = \sigma_{\varphi(g,\alpha')}(x) = \sigma_g(x),$$

taking care of (vii).

The verification of (viii) is done as follows: for $\alpha = \alpha'\alpha''$, as in (2.4.1), one has

$$\sigma^*_1(\alpha) = \sigma^*_1(\alpha'\alpha'') = \sigma_1(\alpha')\sigma^*_{\varphi(1,\alpha')}(\alpha'') \overset{(2.2)}{=} \sigma_1(\alpha')\sigma^*_1(\alpha'') = \alpha'\alpha'' = \alpha.$$

In order to prove (ix), pick paths α in E^k and β in E^l, where $k+l = m+1$, and such that $d(\alpha) = r(\beta)$.

We leave it for the reader to verify (ix) in the easy case in which $k = 0$, that is, when α is a vertex. The case $k = 1$ is also easy as it is nothing but the definition of σ^*_g given in (2.4.1). So we may assume that $k \geq 2$.

Writing $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $\alpha'' \in E^{k-1}$, we then have that $\alpha\beta = \alpha'\alpha''\beta$, and hence, by definition,

$$\sigma^*_g(\alpha\beta) = \sigma_g(\alpha')\sigma^*_{\varphi(g,\alpha')}(\alpha''\beta) = \sigma_g(\alpha')\sigma^*_{\varphi(g,\alpha')}(\alpha'')\ \sigma^*_{\varphi^*(\varphi(g,\alpha'),\alpha'')}(\beta) =$$
$$= \sigma^*_g(\alpha'\alpha'')\ \sigma^*_{\varphi^*(g,\alpha'\alpha'')}(\beta).$$

We remark that, in last step above, one should use the induction hypothesis in case $k \leq m$, and the definitions of σ^* and φ^*, when $k = m + 1$.

To verify (x) we again pick paths α in E^k and β in E^l, where $k + l = m + 1$, and such that $d(\alpha) = r(\beta)$. We once more leave the easy case $k = 0$ to the reader and observe that the case $k = 1$ follows from the definition of φ^*.

We may then suppose that $k \geq 2$, so we write $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $\alpha'' \in E^{k-1}$. Then

$$\varphi^*(g, \alpha\beta) = \varphi^*(g, \alpha'\alpha''\beta) = \varphi^*\big(\varphi(g, \alpha'), \alpha''\beta\big) = \varphi^*\Big(\varphi^*\big(\varphi(g, \alpha'), \alpha''\big), \beta\Big) =$$
$$= \varphi^*\Big(\varphi^*(g, \alpha'\alpha''), \beta\Big) = \varphi^*\Big(\varphi^*(g, \alpha), \beta\Big).$$

Let us now prove that σ^* is in fact an action of G on E^n. We begin by proving that $\sigma^*_g\sigma^*_h = \sigma^*_{gh}$ on E^n, for every g and h in G, which we do by induction on n.

This follows immediately from the hypothesis for $n \leq 1$, so let us assume that $n \geq 2$. Given $\alpha \in E^n$, write $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $\alpha'' \in E^{n-1}$. Then

$$\sigma^*_g\big(\sigma^*_h(\alpha)\big) = \sigma^*_g\big(\sigma^*_h(\alpha'\alpha'')\big) = \sigma^*_g\big(\sigma_h(\alpha')\sigma_{\varphi(h,\alpha')}(\alpha'')\big) =$$
$$= \sigma_g\big(\sigma_h(\alpha')\big)\sigma^*_{\varphi(g,\sigma_h(\alpha'))}\big(\sigma_{\varphi(h,\alpha')}(\alpha'')\big) = \sigma_{gh}(\alpha')\sigma^*_{\varphi(g,\sigma_h(\alpha'))\varphi(h,\alpha')}(\alpha'') =$$
$$= \sigma_{gh}(\alpha')\sigma^*_{\varphi(gh,\alpha')}(\alpha'') = \sigma^*_{gh}(\alpha'\alpha'') = \sigma^*_{gh}(\alpha).$$

That α^*_g is bijective on each E^n then follows[2] from (viii), so α^* is indeed an action of G on E^n.

Finally, let us show that φ^* is a cocycle for σ^* on E^n. For this fix g and h in G and let $\alpha \in E^n$. Then, with $\alpha = \alpha'\alpha''$, as before,

$$\varphi^*(gh, \alpha) = \varphi^*(gh, \alpha'\alpha'') = \varphi^*(\varphi(gh, \alpha'), \alpha'') = \varphi^*\Big(\varphi\big(g, \sigma_h(\alpha')\big)\varphi(h, \alpha'), \alpha''\Big) =$$
$$= \varphi^*\Big(\varphi\big(g, \sigma_h(\alpha')\big), \sigma^*_{\varphi(h,\alpha')}(\alpha'')\Big)\varphi^*\big(\varphi(h, \alpha'), \alpha''\big) =: (\star).$$

On the other hand, focusing on the right-hand-side of (2.1), notice that

$$\varphi^*(g, \sigma^*_h(\alpha))\varphi^*(h, \alpha) = \varphi^*\big(g, \sigma^*_h(\alpha'\alpha'')\big)\varphi^*(h, \alpha'\alpha'') =$$
$$= \varphi^*\big(g, \sigma_h(\alpha')\sigma^*_{\varphi(h,\alpha')}(\alpha'')\big)\varphi^*\big(\varphi(h, \alpha'), \alpha''\big) =$$
$$= \varphi^*\Big(\varphi\big(g, \sigma_h(\alpha')\big), \sigma^*_{\varphi(h,\alpha')}(\alpha'')\Big)\varphi^*\big(\varphi(h, \alpha'), \alpha''\big),$$

which coincides with $(\star)$ above. This concludes the proof. $\square$

[2] This is why it is useful to include (viii) as a separate statement, since we may now use it to prove bijectivity.

The only action of G on E^* to be considered in this paper is σ^* so, from now on, we will adopt the shorthand notation

$$g\alpha = \sigma^*_g(\alpha).$$

Moreover, since φ^* extends φ, we will drop the star decoration and denote φ^* simply as φ. The group law, the cocycle condition, and properties (ii, v, vi, vii, ix, x) of (2.4) may then be rewritten as follows:

2.5. Equations. *For every g and h in G, for every $x \in E^0$, and for every α and β in E^* such that $d(\alpha) = r(\beta)$, one has that*

(a) $(gh)\alpha = g(h\alpha)$,

(b) $\varphi(gh, \alpha) = \varphi\big(g, h\alpha\big)\varphi(h, \alpha)$,

(ii) $\varphi(g, x) = g$,

(v) $r(g\alpha) = gr(\alpha)$,

(vi) $d(g\alpha) = gd(\alpha)$,

(vii) $\varphi(g, \alpha)x = gx$,

(ix) $g(\alpha\beta) = (g\alpha)\ \varphi(g, \alpha)\beta$,

(x) $\varphi(g, \alpha\beta) = \varphi\big(\varphi(g, \alpha), \beta\big)$.

It might be worth noticing that if $\varphi(g, \alpha) = 1$, then (2.5.ix) reads "$g(\alpha\beta) = (g\alpha)\beta$", which may be viewed as an associativity property. However associativity does not hold in general as φ is not always trivial, and hence parentheses must be used.

On the other hand parentheses are unnecessary in expressions of the form $\alpha g\beta$, when $\alpha, \beta \in E^*$, and $g \in G$, since the only possible interpretation for this expression is the concatenation of α with $g\beta$.

Another useful property of φ is in order.

2.6. Proposition. *For every $g \in G$, and every $\alpha \in E^*$, one has that*

$$\varphi(g^{-1}, \alpha) = \varphi(g, g^{-1}\alpha)^{-1}.$$

Proof. We have

$$1 = \varphi(1, \alpha) = \varphi(gg^{-1}, \alpha) = \varphi(g, g^{-1}\alpha)\varphi(g^{-1}, \alpha),$$

from where the conclusion follows. □

3. The universal C*-algebra $\mathcal{O}_{G,E}$.

As in the above section we fix a graph E, an action of a group G on E, and a one-cocycle φ satisfying (2.3).

It is our next goal to build a C*-algebra from this data but first let us recall the following notion from [**15**]:

3.1. Definition. A Cuntz-Krieger E-family consists of a set

$$\{p_x : x \in E^0\}$$

of mutually orthogonal projections and a set

$$\{s_e : e \in E^1\}$$

of partial isometries satisfying

(i) $s_e^* s_e = p_{d(e)}$, for every $e \in E^1$.

(ii) $p_x = \sum_{e \in r^{-1}(x)} s_e s_e^*$, for every $x \in E^0$ for which $r^{-1}(x)$ is finite and nonempty.

3.2. Definition. We define $\mathcal{O}_{G,E}$ to be the universal unital C*-algebra generated by a set

$$\{p_x : x \in E^0\} \cup \{s_e : e \in E^1\} \cup \{u_g : g \in G\},$$

subject to the following relations:

(a) $\{p_x : x \in E^0\} \cup \{s_e : e \in E^1\}$ is a Cuntz-Krieger E-family,

(b) the map $u : G \to \mathcal{O}_{G,E}$, defined by the rule $g \mapsto u_g$, is a unitary representation of G,

(c) $u_g s_e = s_{ge} u_{\varphi(g,e)}$, for every $g \in G$, and $e \in E^1$,

(d) $u_g p_x = p_{gx} u_g$, for every $g \in G$, and $x \in E^0$.

Observe that, under our standing assumptions (2.3), for every $x \in E^0$ we have that $r^{-1}(x)$ is finite and nonempty. So (3.1.ii) and (3.2.a) imply that

$$u_g p_x u_g^* = \sum_{r(e)=x} u_g s_e s_e^* u_g^* = \sum_{r(e)=x} s_{ge} u_{\varphi(g,e)} u_{\varphi(g,e)}^* s_{ge}^* =$$

$$= \sum_{r(e)=x} s_{ge} s_{ge}^* = \sum_{r(f)=gx} s_f s_f^* = p_{gx},$$

which says that (3.2.d) follows from the other conditions. We have nevertheless included it in (3.2) in the belief that our theory may be generalized to graphs with sources.

Our construction generalizes some well known constructions in the literature as we would now like to mention.

3.3. Example. Let (G, X) be a self similar group as in [**11**: Definition 2.1]. We may then consider a graph E having only one vertex and such that $E^1 = X$. If we define

$$\varphi(g, x) = g|_x,$$

where, in the terminology of [**11**], $g|_x$ is the restriction (or section) of g at x, then the triple (G, E, φ) satisfies (2.3) and one may show that $\mathcal{O}_{G,E}$ is isomorphic to the algebra $\mathcal{O}_{(G,X)}$ introduced by Nekrashevych in [**11**].

3.4. Example. As in [**7**], given a positive integer N, let $A \in M_N(\mathbb{Z}^+)$ without zero rows and let $B \in M_N(\mathbb{Z})$ be such that

$$A_{i,j} = 0 \quad \Rightarrow \quad B_{i,j} = 0.$$

Consider the graph E with

$$E^0 = \{1, 2, \ldots, N\},$$

and whose adjacency matrix is A. For each pair of vertexes $i, j \in E^0$, such that $A_{i,j} \neq 0$, denote the set of edges with range i and source j by

$$\{e_{i,j,n} : 0 \leq n < A_{i,j}\}.$$

Define an action σ of $\mathbb{Z}$ on E, which is the identity on E^0, and which acts on edges as follows: given $m \in \mathbb{Z}$, and $e_{i,j,n} \in E^1$, let $(\hat{k}, \hat{n})$ be the unique pair of integers such that

$$mB_{i,j} + n = \hat{k}A_{i,j} + \hat{n}, \quad \text{and} \quad 0 \leq \hat{n} < A_{i,j}.$$

Thus, $\hat{k}$ is the quotient and $\hat{n}$ is the remainder of the division of $mB_{i,j} + n$ by $A_{i,j}$. We then put

$$\sigma_m(e_{i,j,n}) = e_{i,j,\hat{n}}.$$

In other words, σ_m corresponds to the addition of $mB_{i,j}$ to the variable "n" of "$e_{i,j,n}$", taken modulo $A_{i,j}$. In turn, the one-cocycle is defined by

$$\varphi(m, e_{i,j,n}) = \hat{k}.$$

It may be proved without much difficulty that $\mathcal{O}_{G,E}$ is isomorphic to Katsura's [**7**] algebra $\mathcal{O}_{A,B}$, under an isomorphism sending each u_m to the m^{th} power of the unitary

$$u := \sum_{i=1}^{N} u_i$$

in $\mathcal{O}_{A,B}$, and sending $s_{e_{i,j,n}}$ to $s_{i,j,n}$.

When $N = 1$, the relevant graph for Katsura's algebras is the same as the one we used above in the description of Nekrashevych's example. However the former is not a special case of the latter because, contrary to what is required in [**11**], the group action might not be faithful.

We now return to the general case of a triple (G, E, φ) satisfying (2.3). We initially recall the usual extension of the notation "s_e" to allow for paths of arbitrary length.

3.5. Definition. *Given a finite path α in E^*, we shall let s_α denote the element of $\mathcal{O}_{G,E}$ given by:*

(i) *when $\alpha = x \in E^0$, we let $s_\alpha = p_x$,*
(ii) *when $\alpha \in E^1$, then s_α is already defined above,*
(iii) *when $\alpha \in E^n$, with $n > 1$, write $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $\alpha'' \in E^{n-1}$, and set $s_\alpha = s_{\alpha'}s_{\alpha''}$, by recurrence.*

Commutation relation (3.2.c) may then be generalized to finite paths as follows:

3.6. Lemma. *Given $\alpha \in E^*$ and $g \in G$, one has that*

$$u_g s_\alpha = s_{g\alpha} u_{\varphi(g,\alpha)}.$$

Proof. Let n be the length of α. When $n = 0, 1$, this follows from (3.2.d&c), respectively. When $n > 1$, write $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $\alpha'' \in E^{n-1}$. Using induction, we then have

$$u_g s_\alpha = u_g s_{\alpha'} s_{\alpha''} = s_{g\alpha'} u_{\varphi(g,\alpha')} s_{\alpha''} = s_{g\alpha'} s_{\varphi(g,\alpha')\alpha''} u_{\varphi(\varphi(g,\alpha'),\alpha'')} =$$

$$= s_{(g\alpha')\varphi(g,\alpha')\alpha''} u_{\varphi(g,\alpha'\alpha'')} = s_{g(\alpha'\alpha'')} u_{\varphi(g,\alpha'\alpha'')} = s_{g\alpha} u_{\varphi(g,\alpha)}. \qquad \square$$

Our next result provides a spanning set for $\mathcal{O}_{G,E}$.

3.7. Proposition. *Let*

$$\mathcal{S} = \big\{ s_\alpha u_g s_\beta^* : \alpha, \beta \in E^*,\ g \in G,\ d(\alpha) = gd(\beta) \big\} \cup \{0\}.$$

Then $\mathcal{S}$ is closed under multiplication and adjoints and its closed linear span coincides with $\mathcal{O}_{G,E}$.

Proof. That $\mathcal{S}$ is closed under adjoints is clear. With respect to closure under multiplication, let $s_\alpha u_g s_\beta^*$ and $s_\gamma u_h s_\delta^*$ be elements of $\mathcal{S}$.

From (3.2.a) we know that $s_\beta^* s_\gamma = 0$, unless either $\gamma = \beta\varepsilon$, or $\beta = \gamma\varepsilon$, for some $\varepsilon \in E^*$. If $\gamma = \beta\varepsilon$, then

$$s_\beta^* s_\gamma = s_\beta^* s_{\beta\varepsilon} = s_\beta^* s_\beta s_\varepsilon = s_\varepsilon,$$

and hence

$$(s_\alpha u_g s_\beta^*)(s_\gamma u_h s_\delta^*) = s_\alpha u_g s_\varepsilon u_h s_\delta^* = s_\alpha s_{g\varepsilon} u_{\varphi(g,\varepsilon)} u_h s_\delta^* = s_{\alpha g\varepsilon} u_{\varphi(g,\varepsilon)h} s_\delta^*. \tag{3.7.1}$$

Moreover, since

$$d(\alpha g\varepsilon) = d(g\varepsilon) = gd(\varepsilon) = \varphi(g,\varepsilon)d(\varepsilon) = \varphi(g,\varepsilon)d(\gamma) = \varphi(g,\varepsilon)hd(\delta),$$

we deduce that the element appearing in the right-hand-side of (3.7.1) indeed belongs to $\mathcal{S}$.

In the second case, namely if $\beta = \gamma\varepsilon$, then the adjoint of the term appearing in the left-hand-side of (3.7.1) is

$$(s_\delta u_{h^{-1}} s_\gamma^*)(s_\beta u_{g^{-1}} s_\alpha^*),$$

and the case already dealt with implies that this belongs to $\mathcal{S}$. The result then follows from the fact that $\mathcal{S}$ is self-adjoint.

In order to prove that $\mathcal{O}_{G,E}$ coincides with the closed linear span of $\mathcal{S}$, let A denote the latter. Given that $\mathcal{S}$ is self-adjoint and closed under multiplication, we see that A is a closed *-subalgebra of $\mathcal{O}_{G,E}$. Since A evidently contains s_α for every α in $E^{\leq 1}$, and that it also contains u_g for every g in G, we deduce that $A = \mathcal{O}_{G,E}$. $\square$

4. The inverse semigroup $\mathcal{S}_{G,E}$.

As before, we keep (2.3) in force.

In this section we will give an abstract description of the set $\mathcal{S}$ appearing in (3.7) as well as its multiplication and adjoint operation. The goal is to construct an inverse semigroup from which we will later recover $\mathcal{O}_{G,E}$.

4.1. Definition. Over the set

$$\mathcal{S}_{G,E} = \big\{(\alpha, g, \beta) \in E^* \times G \times E^* : d(\alpha) = gd(\beta)\big\} \cup \{0\},$$

consider a binary *multiplication* operation defined by

$$(\alpha, g, \beta)(\gamma, h, \delta) = \begin{cases} (\alpha g\varepsilon, \quad \varphi(g,\varepsilon)h, \quad \delta), & \text{if } \gamma = \beta\varepsilon, \\ (\alpha, \ g\varphi(h^{-1},\varepsilon)^{-1}, \ \delta h^{-1}\varepsilon), & \text{if } \beta = \gamma\varepsilon, \\ 0, & \text{otherwise,} \end{cases}$$

and a unary *adjoint* operation defined by

$$(\alpha, g, \beta)^* := (\beta, g^{-1}, \alpha).$$

The subset of $\mathcal{S}_{G,E}$ formed by all elements (α, g, β), with $g = 1$, will be denoted by $\mathcal{S}_E$.

It is easy to see that $\mathcal{S}_E$ is closed under the above operations, and that it is isomorphic to the inverse semigroup generated by the canonical partial isometries in the graph C*-algebra of E.

Let us begin with a simple, but useful result:

4.2. Lemma. *Given (α, g, β) and (γ, h, δ) in $\mathcal{S}_{G,E}$, one has*

$$\beta = \gamma \quad \Rightarrow \quad (\alpha, g, \beta)(\gamma, h, \delta) = (\alpha, gh, \delta).$$

Proof. Focusing on the first clause of (4.1), write $\gamma = \beta\varepsilon$, with $\varepsilon = d(\beta)$. Then

$$(\alpha, g, \beta)(\gamma, h, \delta) = (\alpha gd(\beta), \ \varphi\big(g, d(\beta)\big)h, \ \delta) = (\alpha d(\alpha), \ gh, \ \delta) = (\alpha, gh, \delta). \qquad \square$$

4.3. Proposition. *$\mathcal{S}_{G,E}$ is an inverse semigroup with zero.*

Proof. We leave it for the reader to prove that the above operations are well defined and associative. In order to prove the statement it then suffices [**9**: Theorem 1.1.3] to show that, for all $y, z \in \mathcal{S}_{G,E}$, one has that

(i) $yy^*y = y$, and
(ii) yy^* commutes with zz^*.

Given $y = (\alpha, g, \beta) \in \mathcal{S}_{G,E}$, we have by the above Lemma that

$$yy^*y = (\alpha, g, \beta)(\beta, g^{-1}, \alpha)(\alpha, g, \beta) = (\alpha, 1, \alpha)(\alpha, g, \beta) = (\alpha, g, \beta) = y,$$

proving (i). Notice also that

$$yy^* = (\alpha, 1, \alpha) \tag{4.3.1}$$

is an element of the idempotent semi-lattice of $\mathcal{S}_E$, which is a commutative set because $\mathcal{S}_E$ is an inverse semigroup. Point (ii) above then follows immediately, concluding the proof. $\square$

As seen in (4.3.1), the idempotent semi-lattice of $\mathcal{S}_{G,E}$, henceforth denoted by $\mathcal{E}$, is given by

$$\mathcal{E} = \big\{(\alpha, 1, \alpha) : \alpha \in E^*\big\} \cup \{0\}. \tag{4.4}$$

Evidently $\mathcal{E}$ is also the idempotent semi-lattice of $\mathcal{S}_E$.

For simplicity, from now on we will adopt the short-hand notation

$$e_\alpha = (\alpha, 1, \alpha), \quad \forall\, \alpha \in E^*. \tag{4.5}$$

The following is a standard fact in the theory of graph C*-algebras:

4.6. Proposition. *If $\alpha, \beta \in E^*$, then*

$$e_\alpha e_\beta = \begin{cases} e_\alpha, & \text{if there exists } \gamma \text{ such that } \alpha = \gamma\beta, \\ e_\beta, & \text{if there exists } \gamma \text{ such that } \alpha\gamma = \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that if α and β are in E^*, we say that $\alpha \preceq \beta$, if α is a *prefix* of β, i.e. if there exists $\gamma \in E^*$, such that $\alpha\gamma = \beta$. It therefore follows from (4.6) that

$$e_\alpha \le e_\beta \iff \beta \preceq \alpha. \tag{4.7}$$

Another easy consequence of (4.6) is that, for any two elements $e, f \in \mathcal{E}$, one has that either $e \perp f$, or e and f are comparable. In other words

$$e \Cap f \quad \Rightarrow \quad e \le f, \ \text{ or } \ f \le e. \tag{4.8}$$

Recall that, according to [**2**: Definition 11.1], we say that e *intersects* f, in symbols $e \Cap f$, when $ef \neq 0$.

5. Residual freeness and E*-unitarity.

Again working under (2.3), suppose we are given g in G and α in E^* such that $g\alpha = \alpha$, and $\varphi(g, \alpha) = 1$. Then, by (2.5.ix), we have that

$$g(\alpha\beta) = \alpha\beta,$$

whenever $d(\alpha) = r(\beta)$. Such an element g therefore acts trivially on a large set of finite words.

Occasionally this will be an annoying feature which we would rather avoid, so we make the following:

5.1. Definition. We will say that (G, E, φ) is *residually free* if, whenever $(g, e) \in G \times E^1$, is such that $ge = e$, and $\varphi(g, e) = 1$, then $g = 1$.

This property may be generalized to finite paths:

5.2. Proposition. *Suppose that (G, E, φ) is residually free and that $(g, \alpha) \in G \times E^*$, is such that $g\alpha = \alpha$, and $\varphi(g, \alpha) = 1$, then $g = 1$.*

Proof. Assume that there is a counter-example (g, α) to the statement, which we assume is minimal in the sense that $|\alpha|$ is as small as possible.

To be sure, to say that (g, α) is a counter-example is to say that $g\alpha = \alpha$, $\varphi(g, \alpha) = 1$, and yet $g \neq 1$.

By (2.5.ii), α can't be a vertex, and neither can it be an edge, by hypothesis. So $|\alpha| \geq 2$, and we may then write $\alpha = \beta\gamma$, with $\beta, \gamma \in E^*$, and $|\beta|, |\gamma| < |\alpha|$. Then

$$\beta\gamma = \alpha = g\alpha = g(\beta\gamma) = (g\beta)\varphi(g, \beta)\gamma,$$

whence $\beta = g\beta$, and $\gamma = \varphi(g, \beta)\gamma$, by length considerations. Should $\varphi(g, \beta) = 1$, the pair (g, β) would be a smaller counter-example to the statement, violating the minimality of α. So we have that $\varphi(g, \beta) \neq 1$. In addition,

$$\varphi\big(\varphi(g, \beta), \gamma\big) = \varphi(g, \beta\gamma) = \varphi(g, \alpha) = 1.$$

It follows that $\big(\varphi(g, \beta), \gamma\big)$ is a counter-example to the statement, violating the minimality of α. This is a contradiction and hence no counter-example exists whatsoever, concluding the proof. ☐

An apparently stronger version of residual freeness is in order.

5.3. Proposition. *Suppose that (G, E, φ) is residually free. Then, for all $g_1, g_2 \in G$, and $\alpha \in E^*$, one has that*

$$g_1\alpha = g_2\alpha \quad \wedge \quad \varphi(g_1, \alpha) = \varphi(g_2, \alpha) \quad \Rightarrow \quad g_1 = g_2.$$

Proof. Defining $g = g_2^{-1}g_1$, observe that $g\alpha = \alpha$, and we claim that $\varphi(g, \alpha) = 1$. In fact,

$$\varphi(g, \alpha) = \varphi\big(g_2^{-1}g_1, \alpha\big) = \varphi\big(g_2^{-1}, g_1\alpha\big)\varphi\big(g_1, \alpha\big) \overset{(2.6)}{=} \varphi\big(g_2, g_2^{-1}g_1\alpha\big)^{-1}\varphi(g_1, \alpha) =$$

$$= \varphi(g_2, \alpha)^{-1}\varphi(g_1, \alpha) = 1,$$

so it follows that $g = 1$, which is to say that $g_1 = g_2$. ☐

Recall that an inverse semigroup $\mathcal{S}$ with zero is called E*-unitary, or sometimes 0-E-unitary [**9**: Chapter 9] if, whenever an element $s \in \mathcal{S}$ dominates a nonzero idempotent e, meaning that $se = e$, then s is necessarily also idempotent.

Residual freeness is very closely related to E*-unitary inverse semigroups, as we would now like to show.

5.4. Proposition. *(G, E, φ) is residually free if and only if $\mathcal{S}_{G,E}$ is an E*-unitary inverse semigroup.*

Proof. Assuming that (G, E, φ) is residually free, let $s = (\alpha, g, \beta)$ be in $\mathcal{S}_{G,E}$, and let $e_\gamma = (\gamma, 1, \gamma)$ be a nonzero idempotent in $\mathcal{E}$, such that $e_\gamma \leq s$. It follows that also

$$e_\gamma \leq s^*s = (\beta, 1, \beta),$$

so $\gamma = \beta\varepsilon$, for some $\varepsilon \in E^*$, by (4.6). The relation "$e_\gamma = se_\gamma$" translates into

$$(\gamma, 1, \gamma) = (\alpha, g, \beta)(\gamma, 1, \gamma) = \big(\alpha g\varepsilon, \varphi(g, \varepsilon), \gamma\big), \tag{5.4.1}$$

so

$$\beta\varepsilon = \gamma = \alpha g\varepsilon,$$

which implies that $\varepsilon = g\varepsilon$, and $\beta = \alpha$. If we further notice that (5.4.1) gives $\varphi(g, \varepsilon) = 1$, we may conclude from (5.2) that $g = 1$, and hence that

$$s = (\alpha, g, \beta) = (\beta, 1, \beta)$$

is an idempotent element. This shows that $\mathcal{S}_{G,E}$ is E*-unitary.

In order to prove the converse, let $(g, e) \in G \times E^1$ be such that $ge = e$, and $\varphi(g, e) = 1$. Then

$$(1, g, 1)(e, 1, e) = (ge, \varphi(g, e), e) = (e, 1, e).$$

Therefore the nonzero element $(1, g, 1)$ dominates the idempotent $(e, 1, e)$ and, assuming that $\mathcal{S}_{G,E}$ is E*-unitary, we conclude that $(1, g, 1)$ is idempotent, which is to say that $g = 1$. This proves that (G, E, φ) is residually free. □

6. Tight representations of $\mathcal{S}_{G,E}$.

As before, we keep (2.3) in force.

It is the main goal of this section to show that $\mathcal{O}_{G,E}$ is the universal C*-algebra for tight representations of $\mathcal{S}_{G,E}$.

Recall from (4.6) that $e_\alpha \le e_{d(\alpha)}$, for every $\alpha \in E^*$, so we see that the set

$$\{e_x : x \in E^0\} \tag{6.1}$$

is a *cover* [**2**: Definition 11.5] for $\mathcal{E}$.

6.2. Proposition. *The map*

$$\pi : \mathcal{S}_{G,E} \to \mathcal{O}_{G,E},$$

defined by $\pi(0) = 0$, *and*

$$\pi(\alpha, g, \beta) = s_\alpha u_g s_\beta^*,$$

is a tight [**2**: Definition 13.1] *representation.*

Proof. We leave it for the reader to show that π is in fact multiplicative and that it preserves adjoints.

In order to prove that π is tight, we shall use the characterization given in [**2**: Proposition 11.8], observing that π satisfies condition (i) of [**2**: Proposition 11.7] because, with respect to the cover (6.1), we have that

$$\bigvee_{x \in E^0} \pi(e_x) = \sum_{x \in E^0} \pi(e_x) = \sum_{x \in E^0} p_x = 1,$$

by (3.2.a). So we assume that $\{e_{\alpha_1}, \ldots, e_{\alpha_n}\}$ is a cover for a given e_β, where $\alpha_1, \ldots \alpha_n, \beta \in E^*$, and we need to show that

$$\bigvee_{i=1}^{n} \pi(e_{\alpha_i}) \geq \pi(e_\beta). \tag{6.2.1}$$

In particular, for each i, we have that $e_{\alpha_i} \leq e_\beta$, which says that there exists $\gamma_i \in E^*$ such that $\alpha_i = \beta\gamma_i$.

We shall prove (6.2.1) by induction on the variable

$$L = \min_{1 \leq i \leq n} |\gamma_i|.$$

If $L = 0$, we may pick i such that $|\gamma_i| = 0$, and then necessarily $\gamma_i = d(\beta)$, in which case $\alpha_i = \beta$, and (6.2.1) is trivially true.

Assuming that $L \geq 1$, one sees that $x := d(\beta)$ is not a source, meaning that $r^{-1}(x)$ is nonempty. Write

$$r^{-1}(x) = \{e_1, \ldots, e_k\},$$

and observe that

$$\pi(e_\beta) = s_\beta s_\beta^* = s_\beta p_x s_\beta^* \overset{(3.2.a)}{=} \sum_{j=1}^{k} s_\beta s_{e_j} s_{e_j}^* s_\beta^* = \sum_{j=1}^{k} \pi(e_{\beta e_j}).$$

In order to prove (6.2.1) it is therefore enough to show that

$$\bigvee_{i=1}^{n} \pi(e_{\alpha_i}) \geq \pi(e_{\beta e_j}), \tag{6.2.2}$$

for all $j = 1, \ldots, k$.

Fixing j we claim that $e_{\beta e_j}$ is covered by the set

$$Z = \big\{e_{\alpha_i} : 1 \leq i \leq n,\ e_{\alpha_i} \leq e_{\beta e_j}\big\}.$$

In order to see this let $x \in \mathcal{E}$ be a nonzero element such that $x \leq e_{\beta e_j}$. Then $x \leq e_\beta$, and so $x \Cap e_{\alpha_i}$ for some i. Thus, to prove the claim it is enough to check that e_{α_i} lies in Z. Observe that

$$x e_{\beta e_j} e_{\alpha_i} = x e_{\alpha_i} \neq 0,$$

which implies that $e_{\beta e_j} \Cap e_{\alpha_i}$.

By (4.8) we have that $e_{\beta e_j}$ and e_{α_i} are comparable, so either $\beta e_j \preceq \alpha_i$ or $\alpha_i \preceq \beta e_j$, by (4.7). Since we are under the hypothesis that $L \geq 1$, and hence that

$$|\alpha_i| = |\beta_i| + |\gamma_i| \geq |\beta| + 1 = |\beta e_j|,$$

we must have that $\beta e_j \preceq \alpha_i$, from where we deduce that $e_{\alpha_i} \leq e_{\beta e_j}$, proving our claim.

Employing the induction hypothesis we then deduce that

$$\bigvee_{z \in Z} \pi(z) \geq \pi(e_{\beta e_j}),$$

verifying (6.2.2), and thus concluding the proof. $\square$

We would now like to prove that the representation π above is in fact the *universal* tight representation of $\mathcal{S}_{G,E}$.

6.3. Theorem. *Let A be a unital C^*-algebra and let $\rho : \mathcal{S}_{G,E} \to A$ be a tight representation. Then there exists a unique unital *-homomorphism $\psi : \mathcal{O}_{G,E} \to A$, such that the diagram*

$$\begin{tikzcd} \mathcal{S}_{G,E} \arrow[r, "\pi"] \arrow[dr, "\rho"'] & \mathcal{O}_{G,E} \arrow[d, "\psi"] \\ & A \end{tikzcd}$$

commutes.

Proof. We will initially prove that the elements

$$\tilde{p}_x := \rho(x, 1, x), \qquad \forall x \in E^0,$$

$$\tilde{s}_e := \rho\big(e, 1, d(e)\big), \qquad \forall e \in E^1,$$

$$\tilde{u}_g := \sum_{x \in E^0} \rho(x, g, g^{-1}x), \quad \forall g \in G,$$

satisfy relations (3.2.a–d). Since the e_x (defined in (4.5)) are mutually orthogonal idempotents in $\mathcal{S}_{G,E}$, it is clear that the $\tilde{p}_x$ are mutually orthogonal projections. Evidently the $\tilde{s}_e$ are partial isometries so, in order to check (3.2.a), we must only verify (3.1.i) and (3.1.ii). With respect to the former, let $e \in E^1$. Then

$$\tilde{s}_e^* \tilde{s}_e = \rho\big((d(e), 1, e)(e, 1, d(e)\big) = \rho\big(d(e), 1, d(e)\big) = \tilde{p}_{d(e)},$$

proving (3.1.i). In order to prove (3.1.ii), let x be a vertex such that $r^{-1}(x)$ is nonempty and write

$$r^{-1}(x) = \{e_1, \ldots, e_n\}.$$

Putting $q_i = (e_i, 1, e_i)$, we then claim that the set

$$\{q_1, \ldots, q_n\}$$

is a cover for $q := (x, 1, x)$. In order to prove this we must show that, if the nonzero idempotent f is dominated by q, then $f \Cap q_i$ for some i.

Let $f = (\alpha, 1, \alpha)$ by (4.4) and notice that

$$0 \neq f = fq = (\alpha, 1, \alpha)(x, 1, x).$$

So α and x are comparable, and this can only happen when $x = r(\alpha)$. If $|\alpha| = 0$ then necessarily $\alpha = x$, so $f = q$, and it is clear that $f \Cap q_i$ for all i. On the other hand, if $|\alpha| \geq 1$, we write

$$\alpha = \alpha' \alpha'',$$

with $\alpha' \in E^1$, so that $r(\alpha') = r(\alpha) = x$, and hence $\alpha' = e_i$, for some i. Therefore

$$fq_i = (\alpha, 1, \alpha)(e_i, 1, e_i) = (\alpha, 1, \alpha)(\alpha', 1, \alpha') = (\alpha, 1, \alpha) \neq 0,$$

so $f \Cap q_i$, proving the claim. Since ρ is a tight representation, we deduce that

$$\rho(q) = \bigvee_{i=1}^{n} \rho(q_i),$$

but since the q_i are easily seen to be pairwise orthogonal, their supremum coincides with their sum, whence

$$\tilde{p}_x = \rho(q) = \sum_{i=1}^{n} \rho(q_i) = \sum_{i=1}^{n} \rho(e_i, 1, e_i) =$$

$$= \sum_{i=1}^{n} \rho\big((e_i, 1, d(e_i))\ (d(e_i), 1, e_i)\big) = \sum_{i=1}^{n} \tilde{s}_{e_i} \tilde{s}^*_{e_i},$$

thus verifying (3.1.ii), and hence proving (3.2.a).

With respect to (3.2.b), let us first prove that $\tilde{u}_1 = 1$. Considering the subsets of $\mathcal{E}$ given by

$$X = \varnothing, \quad Y = \varnothing, \quad \text{and} \quad Z = \big\{(x, 1, x) : x \in E^0\big\},$$

notice that, according to [**2**: Definition 11.4], one has that

$$\mathcal{E}^{X,Y} = \mathcal{E},$$

and that Z is a cover for $\mathcal{E}^{X,Y}$, as seen in (6.1). By the tightness condition [**2**: Definition 11.6] we have

$$\bigvee_{z \in Z} \rho(z) \geq \bigwedge_{x \in X} \rho(x) \wedge \bigwedge_{y \in Y} \neg\rho(y).$$

As explained in the discussion following [**2**: Definition 11.6], the right-hand-side above must be interpreted as 1 because X and Y are empty. On the other hand, since the $\rho(z)$ are pairwise orthogonal, the supremum in the left-hand-side above becomes a sum, so

$$1 = \sum_{z \in Z} \rho(z) = \sum_{x \in E^0} \rho(x, 1, x) = \tilde{u}_1.$$

In order to prove that $\tilde{u}$ is multiplicative, let g and h be in G. Then

$$\tilde{u}_g \tilde{u}_h = \sum_{x, y \in E^0} \rho\big((x, g, g^{-1}x)(y, h, h^{-1}y)\big) =$$

$$= \sum_{x \in E^0} \rho\big((x, g, g^{-1}x)(g^{-1}x, h, h^{-1}g^{-1}x)\big) = \sum_{x \in E^0} \rho\big(x, gh, (gh)^{-1}x\big) = \tilde{u}_{gh}.$$

We next claim that $\tilde u_g^* = \tilde u_{g^{-1}}$, for all g in G. To prove it we compute

$$\tilde u_g^* = \sum_{x\in E^0} \rho(x,g,g^{-1}x)^* = \sum_{x\in E^0} \rho(g^{-1}x,g^{-1},x) = \cdots$$

which, upon the change of variables $y = g^{-1}x$, becomes

$$\cdots = \sum_{y\in E^0} \rho(y,g^{-1},gy) = \tilde u_{g^{-1}}.$$

This shows that $\tilde u$ is a unitary representation, verifying (3.2.b). Turning now our attention to (3.2.c), let $g \in G$ and $e \in E$. Then

$$\tilde u_g \tilde s_e = \sum_{x\in E^0} \rho(x,g,g^{-1}x)\,\rho\big(e,1,d(e)\big) = \rho\big(gr(e),g,r(e)\big)\,\rho\big(e,1,d(e)\big) =$$

$$= \rho\big(r(ge)ge,\varphi(g,e),d(e)\big) = \rho\big(ge,\varphi(g,e),d(e)\big) = (\star).$$

On the other hand

$$\tilde s_{ge}\tilde u_{\varphi(g,e)} = \rho\big(ge,1,d(ge)\big)\sum_{x\in E^0} \rho\big(x,\varphi(g,e),\varphi(g,e)^{-1}x\big) =$$

$$= \rho\big(ge,1,d(ge)\big)\,\rho\big(d(ge),\varphi(g,e),g^{-1}d(ge)\big) =$$

$$= \rho\big(ge,1,d(ge)\big)\,\rho\big(d(ge),\varphi(g,e),d(e)\big) = \rho\big(ge,\varphi(g,e),d(e)\big),$$

which coincides with $(\star)$ and hence proves (3.2.c). We leave the proof of (3.2.d) to the reader after which the universal property of $\mathcal{O}_{G,E}$ intervenes to provide us with a *-homomorphism

$$\psi : \mathcal{O}_{G,E} \to A$$

sending

$$p_x \mapsto \tilde p_x, \quad s_e \mapsto \tilde s_e, \quad \text{and} \quad u_g \mapsto \tilde u_g.$$

Now we must show that

$$\psi\big(\pi(\gamma)\big) = \rho(\gamma), \quad \forall\, \gamma \in \mathcal{S}_{G,E}. \tag{6.3.1}$$

We will first do so for the following special cases:

(i) $\gamma = (x,1,x)$, for $x \in E^0$,

(ii) $\gamma = \big(e,1,d(e)\big)$, for $e \in E^1$,

(iii) $\gamma = (x,g,g^{-1}x)$, for $x \in E^0$, and $g \in G$.

In case (i) we have

$$\psi(\pi(\gamma)) = \psi(\pi(x,1,x)) = \psi(p_x) = \tilde p_x = \rho(x,1,x) = \rho(\gamma).$$

As for (ii),

$$\psi(\pi(\gamma)) = \psi\big(\pi\big(e, 1, d(e)\big)\big) = \psi(s_e) = \tilde{s}_e = \rho\big(e, 1, d(e)\big) = \rho(\gamma).$$

Under (iii),

$$\psi(\pi(\gamma)) = \psi\big(\pi(x, g, g^{-1}x)\big) = \psi(p_x u_g p_{g^{-1}x}) = \psi(p_x u_g) = \tilde{p}_x \tilde{u}_g =$$

$$= \rho(x, 1, x) \sum_{y \in E^0} \rho(y, g, g^{-1}y) = \sum_{y \in E^0} \rho\big((x, 1, x)(y, g, g^{-1}y)\big) = \rho(x, g, g^{-1}x) = \rho(\gamma).$$

In order to prove (6.3.1), it is now clearly enough to check that the *-sub-semigroup of $\mathcal{S}_{G,E}$ generated by the elements mentioned in (i–iii), above, coincides with $\mathcal{S}_{G,E}$.

Denoting this *-sub-semigroup by $\mathcal{T}$, we will first show that $\big(\alpha, 1, d(\alpha)\big)$ is in $\mathcal{T}$, for every $\alpha \in E^*$. This is evident for $|\alpha| \le 1$, so we suppose that $\alpha = \alpha'\alpha''$, with $\alpha' \in E^1$, and $r(\alpha'') = d(\alpha')$. We then have by induction that

$$\mathcal{T} \ni \big(\alpha', 1, d(\alpha')\big)\big(\alpha'', 1, d(\alpha'')\big) = \big(\alpha\alpha'', 1, d(\alpha'')\big) = \big(\alpha, 1, d(\alpha)\big).$$

Considering a general element $(\alpha, g, \beta) \in \mathcal{S}_{G,E}$, let $x = d(\alpha)$, so that $g^{-1}x = d(\beta)$, and notice that

$$\mathcal{T} \ni \big(\alpha, 1, d(\alpha)\big)(x, g, g^{-1}x)\big(\beta, 1, d(\beta)\big)^* =$$

$$= \big(\alpha, 1, d(\alpha)\big)\big(d(\alpha), g, d(\beta)\big)\big(d(\beta), 1, \beta\big) = (\alpha, g, \beta),$$

which proves that $\mathcal{T} = \mathcal{S}_{G,E}$, and hence that (6.3.1) holds.

To conclude we observe that the uniqueness of ψ follows from the fact that $\mathcal{O}_{G,E}$ is generated by the p_x, the s_e, and the u_g. ☐

Given an inverse semigroup $\mathcal{S}$ with zero, recall from [**2**: Theorem 13.3] that $\mathcal{G}_{\text{tight}}(\mathcal{S})$ (denoted simply as $\mathcal{G}_{\text{tight}}$ in [**2**]) is the groupoid of germs for the natural action of $\mathcal{S}$ on the space of tight filters over its idempotent semi-lattice. Moreover the C*-algebra of $\mathcal{G}_{\text{tight}}(\mathcal{S})$ is universal for tight representations of $\mathcal{S}$.

6.4. Corollary. *Under the assumptions of (2.3) one has that $\mathcal{O}_{G,E}$ is isomorphic to the C*-algebra of the groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$.*

Proof. Follows from [**2**: Theorem 13.3] and the uniqueness of universal C*-algebras. ☐

We should notice that our requirement that G be countable in (2.3) is only used in the above proof, where the application of [**2**: Theorem 13.3] depends on the countability of $\mathcal{S}_{G,E}$.

We conclude by analyzing the question of Hausdorffness for $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$.

6.5. Proposition. *When (G, E, φ) is residually free, one has that $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is a Hausdorff groupoid.*

Proof. By (5.4) we have that $\mathcal{S}_{G,E}$ is E*-unitary. The result then follows from [**2**: Propositions 6.2 & 6.4]. ☐

7. Corona Groups.

It is our next goal to give a concrete description of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, similar to the description given to the groupoid associated to a row-finite graph in [**8**: Definition 2.3]. The crucial ingredient there is the notion of *tail equivalence with lag*. In this section we will construct a group where our generalized *lag* function will take values.

Let G be a group. Within the infinite cartesian product[3]

$$G^\infty = \prod_{n\in\mathbb{N}} G$$

consider the infinite direct sum

$$G^{(\infty)} = \bigoplus_{n\in\mathbb{N}} G$$

formed by the elements $g = (g_n)_{n\in\mathbb{N}} \in G^\infty$ which are eventually trivial, that is, for which there exists n_0 such that $g_n = 1$, for all $n \geq n_0$. It is clear that $G^{(\infty)}$ is a normal subgroup of G^∞.

7.1. Definition. Given a group G, the *corona* of G is the quotient group

$$\breve{G} = G^\infty / G^{(\infty)}.$$

Consider the *left* and *right shift* endomorphisms of G^∞

$$\lambda, \rho : G^\infty \to G^\infty$$

given for every $\mathrm{g} = (\mathrm{g}_n)_{n\in\mathbb{N}} \in G^\infty$, by

$$\lambda(\mathrm{g})_n = \mathrm{g}_{n+1}, \quad \forall\, n \in \mathbb{N},$$

and

$$\rho(\mathrm{g})_n = \begin{cases} 1, & \text{if } n = 0, \\ \mathrm{g}_{n-1}, & \text{if } n \geq 1. \end{cases}$$

It is readily seen that $G^{(\infty)}$ is invariant under both λ and ρ, so these pass to the quotient providing endomorphisms

$$\breve{\lambda}, \breve{\rho} : \breve{G} \to \breve{G}. \tag{7.2}$$

For every $\mathrm{g} = (\mathrm{g}_n)_{n\in\mathbb{N}} \in G^\infty$, we have that

$$\lambda(\rho(\mathrm{g})) = \mathrm{g}, \quad \text{and} \quad \rho(\lambda(\mathrm{g})) = (1, \mathrm{g}_2, \mathrm{g}_3, \ldots) \equiv \mathrm{g}, \tag{7.3}$$

where we use "$\equiv$" to refer to the equivalence relation determined by the normal subgroup $G^{(\infty)}$. Therefore both $\breve{\lambda}\breve{\rho}$ and $\breve{\rho}\breve{\lambda}$ coincide with the identity, and hence $\breve{\lambda}$ and $\breve{\rho}$ are each other's inverse. In particular, they are both automorphisms of $\breve{G}$.

Iterating $\breve{\rho}$ therefore gives an action of $\mathbb{Z}$ on $\breve{G}$.

7.4. Definition. Given any countable discrete group G, the *lag group* associated to G is the semi-direct product group

$$\breve{G} \rtimes_{\breve{\rho}} \mathbb{Z}.$$

The reason we call this the "lag group" is that it will play a very important role in the next section, as the co-domain for our *lag* function.

[3] For the purposes of this construction we adopt the convention that $\mathbb{N} = \{1, 2, 3, \ldots\}$.

8. The tight groupoid of $\mathcal{S}_{G,E}$.

We would now like to give a detailed description of the groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$. As already mentioned this is the groupoid of germs for the natural action of $\mathcal{S}_{G,E}$ on the space of tight filters over the idempotent semi-lattice $\mathcal{E}$ of $\mathcal{S}_{G,E}$. See [**2**: Section 4] for more details.

Given an infinite word

$$\xi = \xi_1\xi_2\ldots \in E^\infty,$$

and an integer $n \geq 0$, denote by $\xi|_n$ the finite word of length n given by

$$\xi|_n = \begin{cases} \xi_1\xi_2\ldots\xi_n, & \text{if } n \geq 1, \\ r(\xi_1), & \text{if } n = 0. \end{cases}$$

8.1. Proposition. *There is a unique action*

$$(g, \xi) \in G \times E^\infty \mapsto g\xi \in E^\infty$$

of G on E^∞ such that,

$$(g\xi)|_n = g(\xi|_n),$$

for every $g \in G$, $\xi \in E^\infty$, and $n \in \mathbb{N}$.

Proof. Left to the reader. □

Recall from (4.5) that, for any finite word $\alpha \in E^*$, we denote by e_α the idempotent element $(\alpha, 1, \alpha)$ in $\mathcal{E}$. Thus, given an infinite word $\xi \in E^\infty$, we may look at the subset

$$\mathcal{F}_\xi = \{e_{\xi_n} : n \in \mathbb{N}\} \subseteq \mathcal{E},$$

which turns out to be an ultra-filter [**2**: Definition] over $\mathcal{E}$. Denoting the set of all ultra-filters over $\mathcal{E}$ by $\widehat{\mathcal{E}}_\infty$, as in [**2**: Definition 12.8], one may also show [**2**: Proposition 19.11] that the correspondence

$$\xi \in E^\infty \mapsto \mathcal{F}_\xi \in \widehat{\mathcal{E}}_\infty$$

is bijective, and we will use it to identify E^∞ and $\widehat{\mathcal{E}}_\infty$. Furthermore, this correspondence may be proven to be a homeomorphism if E^∞ is equipped with the product topology.

Since E is finite, E^∞ is compact by Tychonov's Theorem, and consequently so is $\widehat{\mathcal{E}}_\infty$. Being the closure of $\widehat{\mathcal{E}}_\infty$ within $\widehat{\mathcal{E}}$ [**2**: Theorem 12.9], the space $\widehat{\mathcal{E}}_{\text{tight}}$ formed by the tight filters therefore necessarily coincides with $\widehat{\mathcal{E}}_\infty$.

Identifying $\widehat{\mathcal{E}}_{\text{tight}}$ with E^∞, as above, we may transfer the canonical action of $\mathcal{S}_{G,E}$ from the former to the latter resulting in the following: to each element $(\alpha, g, \beta) \in \mathcal{S}_{G,E}$, we associate the partial homeomorphism of E^∞ whose domain is the *cylinder*

$$Z(\beta) := \{\eta \in E^\infty : \eta = \beta\xi, \text{ for some } \xi \in E^\infty\},$$

and which sends each $\eta = \beta\xi \in Z(\beta)$ to $\alpha g\xi$, where the meaning of "$g\xi$" is as in (8.1).

As before we will not use any special symbol to indicate this action, using module notation instead:

$$(\alpha, g, \beta)\eta = \alpha g\xi.$$

Before we proceed let us at least check that $\alpha g\xi$ is in fact an element of E^∞, which is to say that $d(\alpha) = r(g\xi)$. Firstly, for every element $(\alpha, g, \beta) \in \mathcal{S}_{G,E}$, we have that $d(\alpha) = gd(\beta)$. Secondly, if $\eta = \beta\xi \in E^\infty$, then $d(\beta) = r(\xi)$. Therefore

$$r(g\xi) = gr(\xi) = gd(\beta) = d(\alpha).$$

This leads to a first, more or less concrete description of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, namely the groupoid of germs for the above action of $\mathcal{S}_{G,E}$ on E^∞. Our aim is nevertheless a much more precise description of it.

Recall from [**2**: Definition 4.6] that the germ of an element $s \in \mathcal{S}_{G,E}$ at a point ξ in the domain of s is denoted by $[s, \xi]$. If $s = (\alpha, g, \beta)$, this would lead to the somewhat awkward notation $[(\alpha, g, \beta), \xi]$, which from now on will be simply written as

$$\big[\alpha, g, \beta;\, \xi\big].$$

Thus the groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, consisting of all germs for the action of $\mathcal{S}_{G,E}$ on E^∞, is given by

$$\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}) = \Big\{\big[\alpha, g, \beta;\, \xi\big] : (\alpha, g, \beta) \in \mathcal{S}_{G,E},\ \xi \in Z(\beta)\Big\}. \tag{8.2}$$

Let us now prove a criterion for equality of germs.

8.3. Proposition. *Suppose that (G, E, φ) is residually free and let us be given elements (α_1, g_1, β_1) and (α_2, g_2, β_2) in $\mathcal{S}_{G,E}$, with $|\beta_1| \leq |\beta_2|$, as well as infinite paths η_1 in $Z(\beta_1)$, and η_2 in $Z(\beta_2)$. Then*

$$\big[\alpha_1, g_1, \beta_1;\, \eta_1\big] = \big[\alpha_2, g_2, \beta_2;\, \eta_2\big]$$

if and only if there is a finite path $\gamma \in E^*$ and an infinite path $\xi \in E^\infty$, such that

(i) $\alpha_2 = \alpha_1 g_1 \gamma$,

(ii) $g_2 = \varphi(g_1, \gamma)$,

(iii) $\beta_2 = \beta_1 \gamma$,

(iv) $\eta_1 = \eta_2 = \beta_1 \gamma \xi$.

Proof. Assuming that the germs are equal, we have by [**2**: Definition 4.6] that

$$\eta_1 = \eta_2 =: \eta,$$

and there is an idempotent $(\delta, 1, \delta) \in \mathcal{E}$, such that $\eta \in Z(\delta)$, and

$$(\alpha_1, g_1, \beta_1)(\delta, 1, \delta) = (\alpha_2, g_2, \beta_2)(\delta, 1, \delta). \tag{8.3.1}$$

It follows that $\eta = \delta\zeta$, for some $\zeta \in E^\infty$. Upon replacing δ by a longer prefix of η, we may assume that $|\delta|$ is as large as we want. Furthermore the element of $\mathcal{S}_{G,E}$ represented by the two sides of the equation displayed above is evidently nonzero because the partial homeomorphism associated to it under our action has η in its domain. So, focusing on (4.1), we see that β_1 and δ are comparable, and so are β_2 and δ.

Assuming that $|\delta|$ exceeds both $|\beta_1|$ and $|\beta_2|$, we may then write $\delta = \beta_1\varepsilon_1 = \beta_2\varepsilon_2$, for suitable ε_1 and ε_2 in E^*. But since $|\beta_1| \leq |\beta_2|$, this in turn implies that $\beta_2 = \beta_1\gamma$, for some $\gamma \in E^*$, hence proving (iii). Therefore $\delta = \beta_1\gamma\varepsilon_2$, so

$$\eta = \delta\zeta = \beta_1\gamma\varepsilon_2\zeta,$$

and (iv) follows once we choose $\xi = \varepsilon_2\zeta$. Moreover, equation (8.3.1) reads

$$(\alpha_1, g_1, \beta_1)(\beta_1\gamma\varepsilon_2, 1, \beta_1\gamma\varepsilon_2) = (\alpha_2, g_2, \beta_1\gamma)(\beta_1\gamma\varepsilon_2, 1, \beta_1\gamma\varepsilon_2).$$

Computing the products according to (4.1), we get

$$\big(\alpha_1 g_1(\gamma\varepsilon_2),\ \varphi(g_1, \gamma\varepsilon_2),\ \beta_1\gamma\varepsilon_2\big) = \big(\alpha_2 g_2\varepsilon_2,\ \varphi(g_2, \varepsilon_2),\ \beta_1\gamma\varepsilon_2\big),$$

from where we obtain

$$\alpha_2 g_2 \varepsilon_2 = \alpha_1 g_1(\gamma\varepsilon_2) = \alpha_1(g_1\gamma)\varphi(g_1, \gamma)\varepsilon_2, \tag{8.3.2}$$

and

$$\varphi(g_2, \varepsilon_2) = \varphi(g_1, \gamma\varepsilon_2) = \varphi\big(\varphi(g_1, \gamma), \varepsilon_2\big). \tag{8.3.3}$$

Since $|g_2\varepsilon_2| = |\varepsilon_2| = |\varphi(g_1, \gamma)\varepsilon_2|$, we deduce from (8.3.2) that

$$g_2\varepsilon_2 = \varphi(g_1, \gamma)\varepsilon_2, \tag{8.3.4}$$

and hence also that

$$\alpha_2 = \alpha_1 g_1 \gamma,$$

proving (i). Defining $g = g_2^{-1}\varphi(g_1, \gamma)$, we claim that

$$g\varepsilon_2 = \varepsilon_2, \quad \text{and} \quad \varphi(g, \varepsilon_2) = 1.$$

In view of (8.3.3) and (8.3.4), point (ii) follows from (5.3).

Conversely, assume (i–iv) and let us prove equality of the above germs. Setting $\delta = \beta_1\gamma$, we have by (iv) that

$$\eta := \eta_1 = \eta_2 \in Z(\delta),$$

so it suffices to verify (8.3.1), which the reader could do without any difficulty. $\square$

The above result then says that the typical situation in which an equality of germs takes place is

$$\big[\alpha, g, \beta;\ \beta\gamma\xi\big] = \big[\alpha g\gamma, \varphi(g, \gamma), \beta\gamma;\ \beta\gamma\xi\big].$$

Our next two results are designed to offer convenient representatives of germs.

8.4. Proposition. *Given any germ u, there exists an integer n_0, such that for every $n \geq n_0$,*

(i) *there is a representation of u of the form $u = \big[\alpha_1, g_1, \beta_1;\ \beta_1\xi_1\big]$, with $|\alpha_1| = n$.*

(ii) *there is a representation of u of the form $u = \big[\alpha_2, g_2, \beta_2;\ \beta_2\xi_2\big]$, with $|\beta_2| = n$.*

Proof. Write $u = \big[\alpha, g, \beta;\ \eta\big]$, and choose any $n_0 \geq \max\{|\alpha|, |\beta|\}$. Then, for every $n \geq n_0$ we may write $\eta = \beta\gamma\xi$, with $\gamma \in E^*$, $\xi \in E^\infty$, and such that $|\gamma| = n - |\alpha|$ (resp. $|\gamma| = n - |\beta|$). Therefore

$$u = \big[\alpha, g, \beta;\ \beta\gamma\xi\big] = \big[\alpha g\gamma, \varphi(g,\gamma), \beta\gamma;\ \beta\gamma\xi\big],$$

and we have $|\alpha g\gamma| = |\alpha| + |g\gamma| = |\alpha| + |\gamma| = n$ (resp. $|\beta\gamma| = |\beta| + |\gamma| = n$). □

8.5. Corollary. *Given u_1 and u_2 in $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, such that $(u_1, u_2) \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})^{(2)}$ (that is, such that the multiplication u_1u_2 is allowed or, equivalently, such that $d(u_1) = r(u_2)$), there are representations of u_1 and u_2 of the form*

$$u_1 = \big[\alpha_1, g_1, \alpha_2;\ \alpha_2 g_2\xi\big], \quad \textit{and} \quad u_2 = \big[\alpha_2, g_2, \beta;\ \beta\xi\big],$$

and in this case

$$u_1u_2 = \big[\alpha_1, g_1g_2, \beta;\ \beta\xi\big].$$

Proof. Using (8.4), write

$$u_i = \big[\alpha_i, g_i, \beta_i;\ \beta_i\xi_i\big],$$

with $|\beta_1| = |\alpha_2|$. By virtue of (u_1, u_2) lying in $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})^{(2)}$, we have that

$$\beta_1\xi_1 = (\alpha_2, g_2, \beta_2)(\beta_2\xi_2) = \alpha_2 g_2\xi_2,$$

so in fact $\beta_1 = \alpha_2$, and $\xi_1 = g_2\xi_2$. Then

$$u_1 = \big[\alpha_1, g_1, \beta_1;\ \beta_1\xi_1\big] = \big[\alpha_1, g_1, \alpha_2;\ \alpha_2 g_2\xi_2\big],$$

and it suffices to put $\xi = \xi_2$, and $\beta = \beta_2$.

With respect to the last assertion we have that $u_1u_2 = [s; \beta\xi]$, where s is the element of $\mathcal{S}_{G,E}$ given by

$$s = (\alpha_1, g_1, \alpha_2)(\alpha_2, g_2, \beta) \overset{(4.2)}{=} (\alpha_1, g_1g_2, \beta),$$

concluding the proof. □

Having extended the action of G to the set of infinite words in (8.1), one may ask whether it is possible to do the same for the cocycle φ. The following is an attempt at this which however produces a map taking values in the infinite product G^∞, rather than in G.

8.6. Definition. We will denote by Φ, the map

$$\Phi : G \times E^\infty \to G^\infty$$

defined by the rule

$$\Phi(g, \xi)_n = \varphi(g, \xi|_{n-1}),$$

for $g \in G$, $\xi \in E^\infty$, and $n \geq 1$.

We wish to view Φ as some sort of cocycle but, unfortunately, property (2.5.x) does not hold quite as stated. On the fortunate side, a suitable modification of Φ, involving the left shift endomorphism λ of G^∞, works nicely:

8.7. Proposition. *Let α be a finite word in E^* and let ξ be an infinite word in E^∞ such that $d(\alpha) = r(\xi)$. Then, for every g in G, one has that*

$$\Phi\big(\varphi(g, \alpha), \xi\big) = \lambda^{|\alpha|}\big(\Phi\big(g, \alpha\xi\big)\big)$$

Proof. For all $n \geq 1$, we have

$$\Phi\big(\varphi(g, \alpha), \xi\big)_n = \varphi\big(\varphi(g, \alpha), \xi|_{n-1}\big) = \varphi\big(g, \alpha(\xi|_{n-1})\big) =$$
$$= \varphi\big(g, (\alpha\xi)|_{n-1+|\alpha|}\big) = \lambda^{|\alpha|}\big(\Phi(g, \alpha\xi)\big)_n. \qquad \square$$

Another reason to think of Φ as a cocycle is the following version of the cocycle identity (2.5.b):

8.8. Proposition. *For every $\xi \in E^\infty$, and every $g, h \in G$, we have that*

$$\Phi(gh, \xi) = \Phi\big(g, h\xi\big)\Phi(h, \xi).$$

Proof. We have for all $n \in \mathbb{N}$, that

$$\Phi(gh, \xi)_n = \varphi(gh, \xi|_{n-1}) \overset{(2.5.b)}{=} \varphi\big(g, h(\xi|_{n-1})\big)\varphi(h, \xi|_{n-1}) \overset{(8.1)}{=}$$
$$= \varphi\big(g, (h\xi)|_{n-1}\big)\varphi(h, \xi|_{n-1}) = \Phi\big(g, h\xi\big)_n\Phi(h, \xi)_n. \qquad \square$$

The following elementary fact might perhaps justify the choice of "$n-1$" in the definition of Φ.

8.9. Proposition. *Given $g \in G$, and $\xi \in E^\infty$, one has that*

$$(g\xi)_n = \Phi(g, \xi)_n\, \xi_n.$$

Proof. By (8.1) we have that $(g\xi)|_n = g(\xi|_n)$, so the n^{th} letter of $g\xi$ is also the n^{th} letter of $g(\xi|_n)$. In addition we have that

$$g(\xi|_n) = g(\xi|_{n-1}\xi_n) \overset{(2.5.ix)}{=} g(\xi|_{n-1})\varphi(g, \xi|_{n-1})\xi_n,$$

so

$$(g\xi)_n = \varphi(g, \xi|_{n-1})\xi_n = \Phi(g, \xi)_n\, \xi_n. \qquad \square$$

We now wish to define a homomorphism (also called a one-cocycle) from $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ to the lag group $\breve{G} \rtimes_\rho \mathbb{Z}$, by means of the rule

$$\big[\alpha, g, \beta;\, \beta\xi\big] \mapsto \Big(\rho^{|\alpha|}\big(\Phi(g, \xi)\big), |\alpha| - |\beta|\Big).$$

As it is often the case for maps defined on groupoid of germs, the above tentative definition uses a representative of the germ, so some work is necessary to prove that the definition does not depend on the choice of the representative. The technical part of this task is the content of our next result.

8.10. Lemma. *Suppose that (G, E, φ) is residually free. For each $i = 1, 2$, let us be given (α_i, g_i, β_i) in $\mathcal{S}_{G,E}$, as well as $\eta_i = \beta_i\xi_i \in Z(\beta_i)$. If*

$$\big[\alpha_1, g_1, \beta_1;\, \eta_1\big] = \big[\alpha_2, g_2, \beta_2;\, \eta_2\big],$$

then

$$\rho^{|\alpha_1|}\big(\Phi(g_1, \xi_1)\big) \equiv \rho^{|\alpha_2|}\big(\Phi(g_2, \xi_2)\big)$$

modulo $G^{(\infty)}$.

Proof. Assuming without loss of generality that $|\beta_1| \leq |\beta_2|$, we may use (8.3) to write

$$\alpha_2 = \alpha_1 g_1 \gamma, \quad g_2 = \varphi(g_1, \gamma), \quad \beta_2 = \beta_1\gamma, \quad \text{and} \quad \eta_1 = \eta_2 = \beta_1\gamma\xi,$$

for suitable $\gamma \in E^*$ and $\xi \in E^\infty$. Then necessarily $\xi_1 = \gamma\xi$, and $\xi_2 = \xi$, and

$$\rho^{|\alpha_2|}\big(\Phi(g_2, \xi_2)\big) = \rho^{|\alpha_1|+|\gamma|}\Big(\Phi\big(\varphi(g_1, \gamma_1), \xi\big)\Big) \overset{(8.7)}{=}$$

$$= \rho^{|\alpha_1|}\rho^{|\gamma|}\lambda^{|\gamma|}\big(\Phi\big(g_1, \gamma\xi\big)\big) \overset{(7.3)}{\equiv} \rho^{|\alpha_1|}\big(\Phi\big(g_1, \xi_1\big)\big). \qquad \square$$

► Due to our reliance on (8.3) and (8.10), from now on and until the end of this section we will assume, in addition to (2.3), that (G, E, φ) is residually free.

If g is in G^∞, we will denote by $\breve{g}$ its class in the quotient group $\breve{G}$. Likewise we will denote by $\breve{\Phi}$ the composition of Φ with the quotient map

$$\begin{tikzcd} G \times E^\infty \arrow[r, "\Phi"] \arrow[rr, bend right, "\breve{\Phi}"'] & G^\infty \arrow[r] & \breve{G} \end{tikzcd}$$

from G^∞ to $\breve{G}$. It then follows from the above Lemma that the correspondence

$$\big[\alpha, g, \beta;\, \beta\xi\big] \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}) \mapsto \breve{\rho}^{|\alpha|}\big(\breve{\Phi}(g, \xi)\big) \in \breve{G}$$

is a well defined map. This is an important part of the one-cocycle we are about to introduce.

8.11. Proposition. *The correspondence*

$$\ell : \big[\alpha, g, \beta;\, \beta\xi\big] \mapsto \Big(\breve{\rho}^{|\alpha|}\big(\breve{\Phi}(g, \xi)\big), |\alpha| - |\beta|\Big)$$

gives a well defined map

$$\ell : \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}) \to \breve{G} \rtimes_\rho \mathbb{Z},$$

which is moreover a one-cocycle. From now on ℓ will be called the lag function.

Proof. By the discussion above we have that the first coordinate of the above pair is well defined. On the other hand, in the context of (8.3) one easily sees that $|\alpha_1| - |\beta_1| = |\alpha_2| - |\beta_2|$, so the second coordinate is also well defined.

In order to show that ℓ is multiplicative, pick $(u_1, u_2) \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})^{(2)}$. We may then use (8.5) to write

$$u_1 = \big[\alpha_1, g_1, \alpha_2;\ \alpha_2 g_2 \xi\big], \quad \text{and} \quad u_2 = \big[\alpha_2, g_2, \beta;\ \beta\xi\big].$$

So

$$\ell(u_1)\ell(u_2) = \Big(\rho^{|\alpha_1|}\big(\Phi(g_1, g_2\xi)\big),\ |\alpha_1| - |\alpha_2|\Big)\Big(\rho^{|\alpha_2|}\big(\Phi(g_2, \xi)\big),\ |\alpha_2| - |\beta|\Big) =$$

$$= \Big(\rho^{|\alpha_1|}\big(\Phi(g_1, g_2\xi)\big)\ \rho^{|\alpha_1|}\big(\Phi(g_2, \xi)\big),\ \ |\alpha_1| - |\alpha_2| + |\alpha_2| - |\beta|\Big) =$$

$$= \Big(\rho^{|\alpha_1|}\Big(\Phi(g_1, g_2\xi)\Phi(g_2, \xi)\Big),\ |\alpha_1| - |\beta|\Big) \overset{(8.8)}{=} \Big(\rho^{|\alpha_1|}\big(\Phi(g_1 g_2, \xi)\big),\ |\alpha_1| - |\beta|\Big) =$$

$$= \ell\big(\big[\alpha_1, g_1 g_2, \beta;\ \beta\xi\big]\big) \overset{(8.5)}{=} \ell(u_1 u_2). \qquad \square$$

The main relevance of this one-cocycle is that it essentially describes the elements of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, as we would like to show now.

8.12. Proposition. *Given $u_1, u_2 \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, one has that*

$$d(u_1) = d(u_2) \quad \wedge \quad r(u_1) = r(u_2) \quad \wedge \quad \ell(u_1) = \ell(u_2) \quad \Rightarrow \quad u_1 = u_2.$$

Proof. Using (8.4), write $u_i = \big[\alpha_i, g_i, \beta_i;\ \beta_i\xi_i\big]$, for $i = 1, 2$, with $|\beta_1| = |\beta_2|$. Since

$$\beta_1\xi_1 = d(u_1) = d(u_2) = \beta_2\xi_2,$$

we conclude that $\beta_1 = \beta_2$, and

$$\xi_1 = \xi_2 =: \xi.$$

By focusing on the second coordinate of $\ell(u_i)$, we see that $|\alpha_1| - |\beta_1| = |\alpha_2| - |\beta_2|$, and hence $|\alpha_1| = |\alpha_2|$. Moreover, since

$$\alpha_1 g_1 \xi = \alpha_1 g_1 \xi_1 = r(u_1) = r(u_2) = \alpha_2 g_2 \xi_2 = \alpha_2 g_2 \xi,$$

we see that $\alpha_1 = \alpha_2$, and

$$g_1\xi = g_2\xi. \tag{8.12.1}$$

The fact that $\ell(u_1) = \ell(u_2)$ also implies that

$$\breve{\rho}^{|\alpha_1|}\big(\breve{\Phi}(g_1, \xi)\big) = \breve{\rho}^{|\alpha_2|}\big(\breve{\Phi}(g_2, \xi)\big),$$

and since $\alpha_1 = \alpha_2$, we conclude that $\breve{\Phi}(g_1, \xi) = \breve{\Phi}(g_2, \xi)$, and hence that there exists an integer n_0 such that

$$\varphi(g_1, \xi|_n) = \varphi(g_2, \xi|_n), \quad \forall\, n \geq n_0.$$

By (8.12.1) we also have that $g_1(\xi|_n) = g_2(\xi|_n)$, so (5.3) gives $g_1 = g_2$, whence $u_1 = u_2$. $\square$

As a consequence of the above result we see that the map

$$F : \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}) \to E^\infty \times (\breve{G} \rtimes_{\breve{\rho}} \mathbb{Z}) \times E^\infty$$

defined by the rule

$$F(u) = \big(r(u), \ell(u), d(u)\big), \tag{8.13}$$

is one-to-one.

Observe that the co-domain of F has a natural groupoid structure, being the cartesian product of the lag group $\breve{G} \rtimes_{\breve{\rho}} \mathbb{Z}$ by the graph of the transitive equivalence relation on E^∞.

Putting together (8.11) and (8.12) we may now easily prove:

8.14. Corollary. *F is a groupoid homomorphism (functor), hence establishing an isomorphism from $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ to the range of F.*

The range of F is then the concrete model of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ we are after. But, before giving a detailed description of it, let us make a remark concerning notation: since the co-domain of F is a mixture of cartesian and semi-direct products, the standard notation for its elements would be something like $\big(\eta, (u, p), \zeta\big)$, for $\eta, \zeta \in E^\infty$, $u \in \breve{G}$, and $p \in \mathbb{Z}$. As part of our effort to avoid heavy notation we will instead denote such an element by

$$\big(\eta; u, p; \zeta\big).$$

8.15. Proposition. *The range of F is precisely the subset of $E^\infty \times (\breve{G} \rtimes_{\breve{\rho}} \mathbb{Z}) \times E^\infty$, formed by the elements $(\eta; \breve{\mathrm{g}}, p - q; \zeta)$, where $\eta, \zeta \in E^\infty$, $\mathrm{g} \in G^\infty$, and $p, q \in \mathbb{N}$, are such that, for all $n \geq 1$,*

(i) $\mathrm{g}_{n+p+1} = \varphi(\mathrm{g}_{n+p}, \zeta_{n+q})$,

(ii) $\eta_{n+p} = \mathrm{g}_{n+p}\zeta_{n+q}$.

Proof. Pick a general element $\big[\alpha, g, \beta;\, \beta\xi\big] \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ and, recalling that

$$F(\big[\alpha, g, \beta;\, \beta\xi\big]) = \Big(\alpha g\xi;\ \breve{\rho}^{|\alpha|}\big(\breve{\Phi}(g, \xi)\big),\ |\alpha| - |\beta|;\ \beta\xi\Big), \tag{8.15.1}$$

let $\eta = \alpha g\xi$, $\mathrm{g} = \rho^{|\alpha|}\big(\Phi(g, \xi)\big)$, $p = |\alpha|$, $q = |\beta|$, and $\zeta = \beta\xi$, so that the element depicted in (8.15.1) becomes $(\eta; \breve{\mathrm{g}}, p - q; \zeta)$, and we must now verify (i) and (ii). For all $n \geq 1$, one has that

$$\mathrm{g}_{n+|\alpha|} = \Phi(g, \xi)_n = \varphi(g, \xi|_{n-1}),$$

so

$$\eta_{n+p} = (\alpha g\xi)_{n+|\alpha|} = (g\xi)_n \overset{(8.9)}{=} \varphi(g, \xi|_{n-1})\xi_n = \mathrm{g}_{n+|\alpha|}(\beta\xi)_{n+|\beta|} = \mathrm{g}_{n+p}\zeta_{n+q},$$

proving (ii). Also,

$$\mathrm{g}_{n+p+1} = \mathrm{g}_{n+|\alpha|+1} = \varphi\big(g, \xi|_n\big) = \varphi(g, \xi|_{n-1}\xi_n) = \varphi\big(\varphi(g, \xi|_{n-1}), \xi_n\big) =$$

$$= \varphi\big(\mathrm{g}_{n+|\alpha|}, (\beta\xi)_{n+|\beta|}\big) = \varphi\big(\mathrm{g}_{n+p}, \zeta_{n+q}\big),$$

proving (i) and hence showing that the range of F is a subset of the set described in the statement.

Conversely, pick $\eta, \zeta \in E^\infty$, $\mathrm{g} \in G^\infty$, and $p, q \in \mathbb{N}$ satisfying (i) and (ii), and let us show that the element $(\eta; \breve{\mathrm{g}}, p-q; \zeta)$ lies in the range of F. Let

$$g = \mathrm{g}_{p+1}, \quad \alpha = \eta|_p, \quad \text{and} \quad \beta = \zeta|_q,$$

so $\zeta = \beta\xi$ for a unique $\xi \in E^\infty$. We then claim that $\big[\alpha, g, \beta;\, \beta\xi\big]$ lies in $\mathcal{G}_{\mathrm{tight}}(\mathcal{S}_{G,E})$. In order to see this notice that

$$gd(\beta) = gd(\zeta_q) = gr(\zeta_{q+1}) = r(\mathrm{g}_{p+1}\zeta_{q+1}) \overset{\text{(ii)}}{=} r(\eta_{p+1}) = d(\eta_p) = d(\alpha),$$

so $(\alpha, g, \beta) \in \mathcal{S}_{G,E}$, and therefore $\big[\alpha, g, \beta;\, \beta\xi\big]$ is indeed a member of $\mathcal{G}_{\mathrm{tight}}(\mathcal{S}_{G,E})$. The proof will then be concluded once we show that

$$F\big(\big[\alpha, g, \beta;\, \beta\xi\big]\big) = (\eta; \breve{\mathrm{g}}, p-q; \zeta),$$

which in turn is equivalent to showing that

(a) $\alpha g\xi = \eta$,
(b) $\breve{\rho}^{|\alpha|}\big(\breve{\Phi}(g,\xi)\big) = \breve{\mathrm{g}}$,
(c) $|\alpha| - |\beta| = p - q$,
(d) $\beta\xi = \zeta$.

Before proving these points we will show that

$$\varphi(\mathrm{g}_{p+1}, \xi|_n) = \mathrm{g}_{n+p+1}, \quad \forall\, n \geq 0. \tag{†}$$

This is obvious for $n = 0$. Assuming that $n \geq 1$ and using induction, we have

$$\varphi(\mathrm{g}_{p+1}, \xi|_n) = \varphi(\mathrm{g}_{p+1}, \xi|_{n-1}\xi_n) = \varphi\big(\varphi(\mathrm{g}_{p+1}, \xi|_{n-1}), \xi_n\big) =$$

$$= \varphi(\mathrm{g}_{n+p}, \zeta_{n+q}) \overset{\text{(i)}}{=} \mathrm{g}_{n+p+1},$$

verifying (†).

Addressing (a) we have to prove that $(\alpha g\xi)_k = \eta_k$, for all $k \geq 1$, but given that α is defined to be $\eta|_p$, this is trivially true for $k \leq p$. On the other hand, for $k = n + p$, with $n \geq 1$, we have

$$(\alpha g\xi)_k = (\alpha g\xi)_{n+p} = (g\xi)_n \overset{(8.9)}{=} \varphi(g, \xi|_{n-1})\xi_n =$$

$$= \varphi(\mathrm{g}_{p+1}, \xi|_{n-1})\xi_n \overset{(†)}{=} \mathrm{g}_{n+p}\zeta_{n+q} \overset{\text{(ii)}}{=} \eta_{n+p} = \eta_k,$$

proving (a). Focusing on (b) we have for all $n \geq 1$ that

$$\rho^{|\alpha|}\big(\Phi(g,\xi)\big)_{p+n} = \Phi(g,\xi)_n = \varphi(\mathrm{g}_{p+1}, \xi|_{n-1}) \overset{(†)}{=} \mathrm{g}_{n+p},$$

proving that $\rho^{|\alpha|}\big(\Phi(g,\xi)\big) \equiv \mathrm{g}$, modulo $G^{(\infty)}$, hence taking care of (b). The last two points, namely (c) and (d) are immediate and so the proof is concluded. □

As an immediate consequence we get a very precise description of the algebraic structure of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$:

8.16. Theorem. *Suppose that (G, E, φ) satisfies the conditions of (2.3) and is moreover residually free. Then $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is isomorphic to the sub-groupoid of $E^\infty \times (\breve{G} \rtimes_{\breve{\rho}} \mathbb{Z}) \times E^\infty$ given by*

$$\mathcal{G}_{G,E} = \left\{ \begin{array}{rcl} (\eta; \breve{\mathrm{g}}, p-q; \zeta) & \in & E^\infty \times (\breve{G} \rtimes_{\breve{\rho}} \mathbb{Z}) \times E^\infty : \\ & & \mathrm{g} \in G^\infty,\ p, q \in \mathbb{N}, \\ & & \mathrm{g}_{n+p+1} = \varphi(\mathrm{g}_{n+p}, \zeta_{n+q}), \\ & & \eta_{n+p} = \mathrm{g}_{n+p}\zeta_{n+q},\ \textit{for all } n \geq 1 \end{array} \right\}.$$

Recall from [**8**] that the C*-algebra of every graph is a groupoid C*-algebra for a certain groupoid constructed from the graph, and informally called the groupoid for the *tail equivalence with lag*.

Viewed through the above perspective, our groupoid may also deserve such a denomination, except that the lag is not just an integer as in [**8**], but an element of the lag group $\breve{G} \rtimes_\rho \mathbb{Z}$ precisely described by the lag function ℓ introduced in (8.11).

9. The topology of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$.

It is now time we look at the topological aspects of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$. In fact what we will do is simply transfer the topology of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ over to $\mathcal{G}_{G,E}$ via F. Not surprisingly F will turn out to be an isomorphism of topological groupoids.

Recall from [**2**: Proposition 4.14] that, if S is an inverse semigroup acting on a locally compact Hausdorff topological space X, then the corresponding groupoid of germs, say $\mathcal{G}$, is topologized by means of the basis consisting of sets of the form

$$\Theta(s, U),$$

where $s \in S$, and U is an open subset of X, contained in the domain of the partial homeomorphism attached to s by the given action. Each $\Theta(s, U)$ is in turn defined by

$$\Theta(s, U) = \Big\{ [s, x] \in \mathcal{G} : x \in U \Big\}.$$

See [**2**: 4.12] for more details.

If we restrict the choice of the U's above to a predefined basis of open sets of X, e.g. the collection of all cylinders in E^∞ in the present case, we evidently get the same topology on the groupoid of germs. Therefore, referring to the model of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ presented in (8.2), we see that a basis for its topology consists of the sets of the form

$$\Theta(\alpha, g, \beta; \gamma) := \Big\{ [\alpha, g, \beta;\, \xi] \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}) : \xi \in Z(\gamma) \Big\}, \tag{9.1}$$

where $(\alpha, g, \beta) \in \mathcal{S}_{G,E}$, and $\gamma \in E^*$. We may clearly suppose that $|\gamma| \geq |\beta|$ and, since $\Theta(\alpha, g, \beta; \gamma) = \varnothing$, unless β is a prefix of γ, we may also assume that $\gamma = \beta\varepsilon$, for some $\varepsilon \in E^*$.

In this case, given any $[\alpha, g, \beta;\, \xi] \in \Theta(\alpha, g, \beta; \gamma)$, notice that $\xi \in Z(\gamma)$, and

$$(\alpha, g, \beta)(\gamma, 1, \gamma) = \big(\alpha g\varepsilon, \varphi(g, \varepsilon), \gamma\big),$$

from where one concludes that

$$\big[\alpha, g, \beta;\, \xi\big] = \big[\alpha g\varepsilon, \varphi(g, \varepsilon), \gamma;\, \xi\big],$$

for all $\xi \in Z(\gamma)$, and hence also that

$$\Theta(\alpha, g, \beta; \gamma) = \Theta\big(\alpha g\varepsilon, \varphi(g, \varepsilon), \gamma; \gamma\big).$$

This shows that any set of the form (9.1) coincides with another such set for which $\beta = \gamma$. We may therefore do away with this repetition and redefine

$$\Theta(\alpha, g, \beta) := \Big\{ \big[\alpha, g, \beta;\, \xi\big] \in \mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}) : \xi \in Z(\beta) \Big\}. \tag{9.2}$$

We have therefore shown:

9.3. Proposition. *The collection of all sets of the form $\Theta(\alpha, g, \beta)$, where (α, g, β) range in $\mathcal{S}_{G,E}$, is a basis for the topology of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$.*

We may now give a precise description of the topology of $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$, once it is viewed from the alternative point of view of (8.16):

9.4. Proposition. *For each (α, g, β) in $\mathcal{S}_{G,E}$, the image of $\Theta(\alpha, g, \beta)$ under F coincides with the set*

$$\Omega(\alpha, g, \beta) := \left\{ \begin{array}{ll} (\eta; \breve{\mathrm{g}}, k; \zeta) \in \mathcal{G}_{G,E} : & \eta \in Z(\alpha),\ \mathrm{g} \in G^\infty,\ k = |\alpha| - |\beta|,\ \zeta \in Z(\beta), \\ & \mathrm{g}_{1+|\alpha|} = g, \\ & \mathrm{g}_{n+|\alpha|+1} = \varphi(\mathrm{g}_{n+|\alpha|}, \zeta_{n+|\beta|}), \\ & \eta_{n+|\alpha|} = \mathrm{g}_{n+|\alpha|}\zeta_{n+|\beta|},\ \textit{for all } n \geq 1 \end{array} \right\},$$

and hence the collection of all such sets form the basis for a topology on $\mathcal{G}_{G,E}$, with respect to which the latter is isomorphic to $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ as topological groupoids.

Proof. Left for the reader. □

We may now summarize the main results obtained so far:

9.5. Theorem. *Suppose that (G, E, φ) satisfies the conditions of (2.3) and is moreover residually free. Then $\mathcal{O}_{G,E}$ is *-isomorphic to the C^*-algebra of the groupoid $\mathcal{G}_{G,E}$ described in (8.16), once the latter is equipped with the topology generated by the basis of open sets $\Omega(\alpha, g, \beta)$ described above, for all (α, g, β) in $\mathcal{S}_{G,E}$.*

10. $\mathcal{O}_{G,E}$ as a Cuntz-Pimsner algebra.

Inspired by Nekrashevych's paper [**10**], we will now give a description of $\mathcal{O}_{G,E}$ as a Cuntz-Pimsner algebra [**14**]. With this we will also be able to prove that $\mathcal{O}_{G,E}$ is nuclear and that $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is amenable when G is an amenable group. As before, we will work under the conditions of (2.3).

We begin by introducing the algebra of coefficients over which the relevant Hilbert bimodule, also known as a correspondence, will later be constructed.

Since the action of G on E preserves length (2.4.iv), we see that the set of vertexes of E is G-invariant, so we get an action of G on E^0 by restriction. By dualization G acts on the algebra $C(E^0)$ of complex valued functions[4] on E^0. We may therefore form the crossed-product C*-algebra

$$A = C(E^0) \rtimes G.$$

Since $C(E^0)$ is a unital algebra, there is a canonical unitary representation of G in the crossed product, which we will denote by $\{v_g\}_{g\in G}$.

On the other hand, $C(E^0)$ is also canonically isomorphic to a subalgebra of A and we will therefore identify these two algebras without further warnings.

For each x in E^0, we will denote the characteristic function of the singleton $\{x\}$ by q_x, so that $\{q_x : x \in E^0\}$ is the canonical basis of $C(E^0)$, and thus A coincides with the closed linear span of the set

$$\big\{q_x v_g : x \in E^0,\ g \in G\big\}. \tag{10.1}$$

For later reference, notice that the covariance condition in the crossed product reads

$$v_g q_x = q_{gx} v_g, \quad \forall\, x \in E^0, \quad \forall\, g \in G. \tag{10.2}$$

Our next step is to construct a correspondence over A. In preparation for this we denote by A^e the right ideal of A generated by $q_{d(e)}$, for each $e \in E^1$. In technical terms

$$A^e = q_{d(e)} A.$$

With the obvious right A-module structure, and the inner product defined by

$$\langle y, z\rangle = y^* z, \quad \forall\, y, z \in A^e,$$

one has that A^e is a right Hilbert A-module. Notice that this in not necessarily a full Hilbert module since $\langle A^e, A^e\rangle$ is the two-sided ideal of A generated by $q_{d(e)}$, which might be a proper ideal in some cases.

As already seen in (10.1), A is spanned by the elements of the form $q_x v_g$. Therefore A^e is spanned by the elements of the form $q_{d(e)} q_x v_g$, but, since the q's are mutually orthogonal, this is either zero or equal to $q_{d(e)} v_g$. Therefore we see that

$$A^e = \overline{\text{span}}\{q_{d(e)} v_g : g \in G\}.$$

[4] Notice that, since E^0 is a finite set, $C(E^0)$ is nothing but $\mathbb{C}^{|E^0|}$.

Introducing the right Hilbert A-module which will later be given the structure of a correspondence over A, we define

$$M = \bigoplus_{e \in E^1} A^e.$$

Observe that if x is a vertex which is the source of many edges, say

$$d^{-1}(x) = \{e_1, e_2, \ldots, e_n\},$$

then

$$A^{e_i} = q_{d(e_i)} A = q_x A,$$

for all i, so that $q_x A$ appears many times as a direct summand of M. However these copies of $q_x A$ should be suitably distinguished, according to which edge e_i is being considered.

On the other hand, notice that if $d^{-1}(x) = \varnothing$, then $q_x A$ does not appear among the summands of M, at all.

Addressing the fullness of M, observe that

$$\langle M, M \rangle = \sum_{\substack{x \in E^0 \\ d^{-1}(x) \neq \varnothing}} A q_x A,$$

so, when E has no *sinks*, that is, when $d^{-1}(x)$ is nonempty for every x, one has that M is full.

Given $e \in E^1$, the element $q_{d(e)}$, when viewed as an element of $A^e \subseteq M$, will play a very special role in what follows, so we will give it a special notation, namely

$$t_e := q_{d(e)}. \tag{10.3}$$

There is a small risk of confusion here in the sense that, if $e_1, e_2 \in E^1$ are such that

$$x := d(e_1) = d(e_2),$$

then (10.3) assigns q_x to both t_{e_1} and t_{e_2}. However the coordinate in which q_x appears in t_{e_i} is determined by the corresponding e_i, so if $e_1 \neq e_2$, then $t_{e_1} \neq t_{e_2}$.

In order to completely dispel any confusion, here is the technical definition:

$$t_e = (m_f)_{f \in E^1},$$

where

$$m_f = \begin{cases} q_{d(e)}, & \text{if } f = e, \\ 0, & \text{otherwise.} \end{cases}$$

We should notice that

$$t_e q_{d(e)} = t_e, \tag{10.4}$$

and that any element of M may be written uniquely as

$$\sum_{e \in E^1} t_e y_e, \tag{10.5}$$

where each $y_e \in A^e$.

As the next step in constructing a correspondence over A, we would now like to define a certain *-homomorphism from A to the algebra $\mathcal{L}(M)$ of adjointable linear operators on M. Since A is a crossed product algebra, this will be accomplished once we produce a covariant representation (ψ, V) of the C*-dynamical system $\big(C(E^0), G\big)$. We begin with the group representation V.

10.6. Definition. For each $g \in G$, let V_g be the linear operator on M given by

$$V_g\Big(\sum_{e\in E^1} t_e y_e\Big) = \sum_{e\in E^1} t_{ge} v_{\varphi(g,e)} y_e,$$

whenever $y_e \in A^e$, for each e in E^1.

By the uniqueness in (10.5), it is clear that V_g is well defined.

10.7. Proposition. *Each V_g is a unitary operator in $\mathcal{L}(M)$. Moreover, the correspondence $g \mapsto V_g$ is a unitary representation of G.*

Proof. Let $g \in G$. We begin by claiming that the expression defining V_g above holds true whenever the y_e are in A, and not necessarily restricted to A^e. Since V_g is clearly additive, we only need to check that

$$V_g(t_e y) = t_{ge} v_{\varphi(g,e)} y, \quad \forall\, y \in A.$$

Observing that $t_e = t_e q_{d(e)}$, we have

$$V_g(t_e y) = V_g(t_e q_{d(e)} y) = t_{ge} v_{\varphi(g,e)} q_{d(e)} y =$$
$$= t_{ge} q_{d(\varphi(g,e)e)} v_{\varphi(g,e)} y \overset{(2.5.\text{vii})}{=} t_{ge} q_{d(ge)} v_{\varphi(g,e)} y = t_{ge} v_{\varphi(g,e)} y,$$

proving the claim. One therefore concludes that V_g is right-A-linear.

We next claim that, for all $e, f \in E^1$, one has

$$\langle V_g(t_e), t_f\rangle = \langle t_e, V_{g^{-1}}(t_f)\rangle. \tag{10.7.1}$$

We have

$$\langle V_g(t_e), t_f\rangle = \langle t_{ge} v_{\varphi(g,e)}, t_f\rangle = v^*_{\varphi(g,e)} \langle t_{ge}, t_f\rangle = [ge{=}f]\, v^{-1}_{\varphi(g,e)} q_{d(ge)} =$$
$$= [ge{=}f]\, q_{d(\varphi(g,e)^{-1}ge)} v^{-1}_{\varphi(g,e)} \overset{(2.5.\text{vii})}{=} [ge{=}f]\, q_{d(e)} v^{-1}_{\varphi(g,e)} = (\star).$$

Starting from the right-hand-side of (10.7.1), we have

$$\langle t_e, V_{g^{-1}}(t_f)\rangle = \langle t_e, t_{g^{-1}f} v_{\varphi(g^{-1},f)}\rangle = [e{=}g^{-1}f]\, q_{d(e)} v_{\varphi(g^{-1},f)} =$$
$$= [ge{=}f]\, q_{d(e)} v_{\varphi(g,g^{-1}f)^{-1}} = [ge{=}f]\, q_{d(e)} v^{-1}_{\varphi(g,e)},$$

which agrees with $(\star)$ above, and hence proves claim (10.7.1). If $y, z \in A$, we then have that

$$\langle V_g(t_e y), t_f z\rangle = y^*\langle V_g(t_e), t_f\rangle z = y^*\langle t_e, V_{g^{-1}}(t_f)\rangle z = \langle t_e y, V_{g^{-1}}(t_f z)\rangle,$$

from where one sees that $\langle V_g(\xi), \eta\rangle = \langle \xi, V_{g^{-1}}(\eta)\rangle$, for all $\xi, \eta \in M$, hence proving that V_g is an adjointable operator with $V_g^* = V_{g^{-1}}$.

Let us next prove that

$$V_g V_h = V_{gh}, \quad \forall\, g, h \in G.$$

By A-linearity it is enough to prove that these operators coincide on the set formed by the t_e's, which is a generating set for M. We thus compute

$$V_g\big(V_h(t_e)\big) = V_g\big(t_{he} v_{\varphi(h,e)}\big) = V_g(t_{he}) v_{\varphi(h,e)} =$$
$$= t_{ghe} v_{\varphi(g,he)} v_{\varphi(h,e)} = t_{ghe} v_{\varphi(gh,e)} = V_{gh}(t_e).$$

Since it is evident that V_1 is the identity operator on M we obtain, as a consequence, that $V_g^{-1} = V_{g^{-1}} = V_g^*$, so each V_g is unitary and the proof is concluded. ☐

In order to complete our covariant pair we must now construct a *-homomorphism from $C(E^0)$ to $\mathcal{L}(M)$. With this in mind we give the following:

10.8. Definition. For every x in E^0, let

$$M_x = \bigoplus_{e\in r^{-1}(x)} A^e,$$

which we view as a complemented sub-module of M. In addition, we let Q_x be the orthogonal projection from M to M_x, so that

$$Q_x(t_e y) = [r(e)=x]\, t_e y, \quad \forall\, e \in E^1, \quad \forall\, y \in A. \tag{10.8.1}$$

Observe that the Q_x are pairwise orthogonal projections and that $\sum_{x\in E^0} Q_x = 1$.

10.9. Definition. Let $\psi : C(E^0) \to \mathcal{L}(M)$ be the unique unital *-homomorphism such that

$$\psi(q_x) = Q_x, \quad \forall\, x \in E^0.$$

From our working hypothesis that E has no sources, we see that for every x in E^0, there is some $e \in E^1$ such that $r(e) = x$. So

$$Q_x(t_e) = t_e,$$

whence $Q_x \neq 0$. Consequently ψ is injective.

10.10. Proposition. *The pair (ψ, V) is a covariant representation of the C^*-dynamical system $\big(C(E^0), G\big)$ in $\mathcal{L}(M)$.*

Proof. All we must do is check the covariance condition

$$V_g\psi(y) = \psi\big(\sigma_g(y)\big)V_g, \quad \forall\, g \in G, \quad \forall\, y \in C(E^0),$$

where σ is the name we temporarily give to the action of G on $C(E^0)$. Since $C(E^0)$ is spanned by the q_x, it suffices to consider $y = q_x$, in which case the above identity becomes

$$V_g Q_x = Q_{gx} V_g. \tag{10.10.1}$$

Furthermore M is generated, as an A-module, by the t_e, for $e \in E^1$, so we only need to verify this on the t_e. We have

$$V_g\big(Q_x(t_e)\big) = [r(e)=x]\, V_g(t_e) = [r(e)=x]\, t_{ge} v_{\varphi(g,e)},$$

while

$$Q_{gx}\big(V_g(t_e)\big) = Q_{gx}\big(t_{ge} v_{\varphi(g,e)}\big) = [r(ge)=gx]\, t_{ge} v_{\varphi(g,e)},$$

verifying (10.10.1) and concluding the proof. □

It follows from [**13**: Proposition 7.6.4 and Theorem 7.6.6] that there exists a *-homomorphism

$$\Psi : C(E^0) \rtimes G \to \mathcal{L}(M),$$

such that

$$\Psi(q_x) = Q_x, \quad \forall\, x \in E^0,$$

and

$$\Psi(v_g) = V_g, \quad \forall\, g \in G.$$

Equipped with the left-A-module structure provided by Ψ, we then have that M is a correspondence over A.

For later reference we record here a few useful calculations involving the left-module structure of M.

10.11. Proposition. *Let $g \in G$, $e \in E^1$, and $x \in E^0$. Then*

(i) $v_g t_e = t_{ge} v_{\varphi(g,e)}$,

(ii) $q_x v_g t_e = [r(ge)=x]\, t_{ge} v_{\varphi(g,e)}$.

Proof. We have

$$v_g t_e = \Psi(v_g) t_e = V_g(t_e) = t_{ge} v_{\varphi(g,e)},$$

proving (a). Also

$$q_x v_g t_e = \Psi(q_x)(v_g t_e) = Q_x(t_{ge} v_{\varphi(g,e)}) = [r(ge)=x]\, t_{ge} v_{\varphi(g,e)}. \qquad \square$$

It is our next goal to prove that $\mathcal{O}_{G,E}$ is naturally isomorphic to the Cuntz-Pimsner algebra associated to the correspondence M, which we denote by $\mathcal{O}_M$. As a first step, we identify a certain Cuntz-Krieger E-family.

10.12. Proposition. *The following relations hold within $\mathcal{O}_M$.*

(a) *For every $x \in E^0$, one has that $\sum_{e \in r^{-1}(x)} t_e t_e^* = q_x$.*

(b) $\sum_{e \in E^1} t_e t_e^* = 1$.

(c) *The set $\{q_x : x \in E^0\} \cup \{t_e : e \in E^1\}$ is a Cuntz-Krieger E-family.*

Proof. We first claim that, for every $x \in E^0$, and every $m \in M$, one has that

$$\sum_{e \in r^{-1}(x)} t_e t_e^* m = q_x m.$$

To prove it, it is enough to consider the case in which $m = t_f$, for $f \in E^1$, since these generate M. In this case we have

$$\sum_{e \in r^{-1}(x)} t_e t_e^* t_f = [r(f)=x]\, t_f t_f^* t_f = [r(f)=x]\, t_f \overset{(10.8.1)}{=} Q_x(t_f) = q_x t_f,$$

proving the claim. This says that the pair $\left(q_x, \sum_{e\in r^{-1}(x)} t_e t_e^*\right)$ is a redundancy or, adopting the terminology of [**14**], that the generalized compact operator

$$\sum_{e\in r^{-1}(x)} \Omega_{t_e,t_e}$$

is mapped to $\Psi(q_x)$ via $\Psi^{(1)}$. Therefore

$$q_x = \sum_{e\in r^{-1}(x)} t_e t_e^*,$$

in $\mathcal{O}_M$, proving (a). Point (b) then follows from the fact that $\sum_{x\in E^0} q_x = 1$.

Focusing now on (c), it is evident that $\{q_x : x \in E^0\}$ is a family of mutually orthogonal projections. Moreover, for each $e \in E^1$, we have

$$t_e^* t_e = \langle t_e, t_e\rangle = q_{d(e)},$$

proving (3.1.i) and also that t_e is a partial isometry. Property (3.1.ii) also holds in view of (a), so the proof is concluded. □

10.13. Proposition. *There exists a unique surjective *-homomorphism*

$$\Lambda : \mathcal{O}_{G,E} \to \mathcal{O}_M$$

such that $\Lambda(p_x) = q_x$, $\Lambda(s_e) = t_e$, *and* $\Lambda(u_g) = v_g$.

Proof. By the universal property of $\mathcal{O}_{G,E}$, in order to prove the existence of Λ it is enough to check that the q_x, t_e, and v_g satisfy the conditions of (3.2).

Condition (3.2.a) has already been proved above while (3.2.b) is evidently true since v is a representation of G in $C(E^0) \rtimes G \subseteq \mathcal{O}_M$. Condition (3.2.c) is precisely (10.11.i), while (3.2.d) was taken care of in (10.2).

Since A is spanned by the q_x and the v_g by (10.1), and since M is generated over A by the t_e, we see that $\mathcal{O}_M$ is spanned by the set

$$\{q_x, t_e, v_g : x \in E^0,\ e \in E^1,\ g \in G\},$$

so Λ is surjective. □

Let us now prove that Λ is invertible by providing an inverse to it. Since A is the crossed product C*-algebra $C(E^0) \rtimes G$, one sees that (3.2.a&d) guarantees the existence of a *-homomorphism

$$\theta_A : A \to \mathcal{O}_{G,E},$$

sending the q_x to the p_x, and the v_g to the u_g. For each e in E^1, consider the linear mapping

$$\theta_M : M \to \mathcal{O}_{G,E},$$

given, for every $m = (m_e)_{e\in E^1} \in M$, by

$$\theta_M(m) = \sum_{e\in E^1} s_e\theta_A(m_e) \in \mathcal{O}_{G,E}.$$

Notice that $\theta_M(t_e) = s_e$, for all $e \in E^1$, because

$$\theta_M(t_e) = s_e\theta_A(q_{d(e)}) = s_e p_{d(e)} = s_e.$$

10.14. Lemma. *The pair (θ_A, θ_M) is a representation of the correspondence M in the sense of* [**14**: Theorem 3.4], *meaning that for all $y \in A$ and all $\xi, \xi' \in M$,*

(i) $\theta_M(\xi)\theta_A(y) = \theta_M(\xi y)$,
(ii) $\theta_A(y)\theta_M(\xi) = \theta_M(y\xi)$,
(iii) $\theta_M(\xi)^*\theta_M(\xi') = \theta_A(\langle\xi, \xi'\rangle)$.

Proof. Considering the various spanning sets at our disposal, we may assume that $y \overset{(10.1)}{=} q_x v_g$, that $\xi = t_e z$, and $\xi' = t_{e'} z'$, with $x \in E^0$, $g \in G$, $e, e' \in E^1$, $z \in q_{d(e)}A$, and $z' \in q_{d(e')}A$. Then

$$\theta_M(\xi)\theta_A(y) = \theta_M(t_e z)\theta_A(y) = s_e\theta_A(z)\theta_A(y) = s_e\theta_A(zy) = \theta_M(t_e zy) = \theta_M(\xi y),$$

proving (i). As for (ii), we have

$$\theta_A(y)\theta_M(\xi) = \theta_A(q_x v_g)\theta_M(t_e z) = p_x u_g s_e\theta_A(z) = p_x s_{ge} u_{\varphi(g,e)}\theta_A(z) =$$

$$= [{\scriptstyle r(ge)=x}]\, s_{ge}\theta_A(v_{\varphi(g,e)}z) = [{\scriptstyle r(ge)=x}]\, \theta_M(t_{ge}v_{\varphi(g,e)}z) \overset{(10.11.\mathrm{ii})}{=} \theta_M(q_x v_g t_e z) = \theta_M(y\xi),$$

proving (ii). Focusing now on (iii), we have

$$\theta_M(\xi)^*\theta_M(\xi') = \big(s_e\theta_A(z)\big)^* s_{e'}\theta_A(z') = [{\scriptstyle e=e'}]\, \theta_A(z)^* p_{d(e)}\theta_A(z') =$$

$$= [{\scriptstyle e=e'}]\, \theta_A(z^* q_{d(e)} z') = \theta_A(\langle\xi, \xi'\rangle). \qquad \square$$

It is well known [**14**: Theorem 3.4] that the Toeplitz algebra for the correspondence M, usually denoted $\mathcal{T}_M$, is universal for representations of M, so there exists a *-homomorphism

$$\Theta_0 : \mathcal{T}_M \to \mathcal{O}_{G,E},$$

coinciding with θ_A on A and with θ_M on M.

10.15. Theorem. *The map Θ_0, defined above, factors through $\mathcal{O}_M$, providing a *-isomorphism*

$$\Theta : \mathcal{O}_M \to \mathcal{O}_{G,E},$$

such that $\Theta(q_x) = p_x$, $\Theta(t_e) = s_e$, and $\Theta(v_g) = u_g$, for all $x \in E^0$, $e \in E^1$, and $g \in G$.

Proof. The factorization property follows immediately from (10.12.b) and an easy modification of [**4**: Proposition 7.1] to Cuntz-Pimsner algebras.

In order to prove that Θ is an isomorphism, observe that $\Theta \circ \Lambda$ coincides with the identity map on the generators of $\mathcal{O}_{G,E}$, by (10.13), and hence $\Theta \circ \Lambda = id$. The result then follows from the fact that Λ is surjective. $\square$

10.16. Corollary. *If G is amenable then $\mathcal{O}_{G,E}$ is nuclear.*

Proof. The amenability of G ensures that $C(E^0) \rtimes G$ is nuclear. The result then follows from (10.15), the fact that Toeplitz-Pimsner algebras over nuclear coefficient algebras is nuclear [**1**: Theorem 4.6.25], and so are quotients of nuclear algebras [**1**: Theorem 9.4.4]. $\square$

10.17. Remark. Since E^0 is finite, the nuclearity of $C(E^0) \rtimes G$ is equivalent to the amenability of G. However, if the present construction is generalized for infinite graphs, one could produce examples of non amenable groups acting amenably on E^0, in which case $C(E^0) \rtimes G$ would be amenable. The proof of (10.16) could then be adapted to prove that $\mathcal{O}_{G,E}$ is nuclear.

10.18. Corollary. *If G is amenable and (G, E, φ) is residually free then $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ and its sibling $\mathcal{G}_{G,E}$ are amenable groupoids.*

Proof. Follows from (10.16), (9.5), and [**1**: Theorem 5.6.18]. $\square$

Nekrashevych has proven in [**11**: Theorem 5.6], that a certain groupoid of germs, denoted $\mathcal{D}_G$, constructed in the context of self-similar groups, is amenable under the hypothesis that the group is *contracting* and *self-replicating*. Even though there are numerous differences between $\mathcal{D}_G$ and $\mathcal{G}_{G,E}$, including a different notion of *germs* and Nekrashevych's requirement that group actions be *faithful*, we believe it should be interesting to try to generalize Nekrashevych's result to our context.

References

[1] N. P. Brown and N. Ozawa, "C*-algebras and finite-dimensional approximations", Graduate Studies in Mathematics, 88, American Mathematical Society, 2008.

[2] R. Exel, "Inverse semigroups and combinatorial C*-algebras", *Bull. Braz. Math. Soc.* **39** (2008), no. 2, 191–313.

[3] R. Exel and E. Pardo, "Representing Kirchberg algebras as inverse semigroup crossed products", preprint, 2013.

[4] R. Exel and A. Vershik, "C*-algebras of irreversible dynamical systems", *Canadian Mathematical Journal*, **58** (2006), 39–63.

[5] R. I. Grigorchuk, "On Burnsides problem on periodic groups", Funct. Anal. Appl., **14** (1980), 41–43.

[6] N. D. Gupta and S. N. Sidki, "On the Burnside problem for periodic groups", Math. Z., **182** (1983), 385–388.

[7] T. Katsura, "A construction of actions on Kirchberg algebras which induce given actions on their K-groups", *J. reine angew. Math.*, **617** (2008), 27–65.

[8] A. Kumjian, D. Pask, I. Raeburn and J. Renault, "Graphs, groupoids, and Cuntz-Krieger algebras", *J. Funct. Anal.,* **144** (1997), 505–541.

[9] M. V. Lawson, "Inverse semigroups, the theory of partial symmetries", World Scientific, 1998.

[10] V. Nekrashevych, "Cuntz-Pimsner algebras of group actions", *J. Operator Theory*, **52** (2004), 223–249.

[11] V. Nekrashevych, "C*-algebras and self-similar groups", *J. reine angew. Math.*, **630** (2009), 59–123.

[12] A. L. T. Paterson, "Groupoids, inverse semigroups, and their operator algebras", Birkhauser, 1999.

[13] G. K. Pedersen, "C*-algebras and Their Automorphism Groups", Academic Press, 1979.

[14] M. V. Pimsner, "A class of C*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$", *Fields Inst. Commun.,* **12** (1997), 189–212.

[15] I. Raeburn, "Graph algebras", CBMS Regional Conference Series in Mathematics, **103** (2005), pp. vi+113.

Departamento de Matemática; Universidade Federal de Santa Catarina; 88010-970 Florianópolis SC; Brazil
(ruyexel@gmail.com)

Departamento de Matemáticas, Facultad de Ciencias; Universidad de Cádiz, Campus de Puerto Real; 11510 Puerto Real (Cádiz); Spain
(enrique.pardo@uca.es)